

Reflected BSDEs with regulated trajectories

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Abstract

We consider reflected backward stochastic different equations with optional barrier and so-called regulated trajectories, i.e trajectories with left and right finite limits. We prove existence and uniqueness results. We also show that the solution may be approximated by a modified penalization method. Application to an optimal stopping problem is given.

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1 Introduction

In the present paper we consider reflected backward stochastic differential equations (RBSDEs for short) with Brownian filtration, one barrier and L^p -data, $p \in [1, 2]$. The main novelty is that we only assume that the barrier is optional. As a consequence the solutions of these equations need not be càdlàg, but are so-called regulated processes, i.e. processes whose trajectories have left and right finite limits. Our motivation for studying such general equations comes from the optimal stopping theory (see [5, 8, 16, 17]).

Let B be a standard d -dimensional Brownian motion and let $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the standard augmentation of the natural filtration generated by B . Suppose we are given an \mathbb{F} -optional process $L = \{L_t, t \in [0, T]\}$, an \mathbb{F} -adapted locally bounded variation process $V = \{V_t, t \in [0, T]\}$, an \mathcal{F}_T -measurable random variable ξ such that $\xi \geq L_T$ (the terminal value) and a measurable function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ (coefficient). In the paper we consider RBSDEs with barrier L of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dK_s + \int_t^T dV_s - \int_t^T Z_s dB_s, \quad t \in [0, T]. \quad (1.1)$$

Roughly speaking, by a solution to (1.1) we understand a triple (Y, Z, K) of \mathbb{F} -progressively measurable processes such that (1.1) is satisfied, Y has regulated trajectories,

$$Y_t \geq L_t, \quad t \in [0, T], \quad (1.2)$$

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and K is an increasing process such $K_0 = 0$ satisfying some minimality condition (see (1.5) below). In case L is càdlàg this condition reads

$$\int_0^T (Y_{t-} - L_{t-}) dK_t = 0. \quad (1.3)$$

An important known result is (see [10]) that for càdlàg barrier the solution (Y, Z, K) of (1.1)–(1.3) leads to the solution of the following optimal stopping problem

$$Y_t = \operatorname{ess\,sup}_{\tau \in \Gamma_t} E \left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t \right), \quad (1.4)$$

where Γ_t is the set of all \mathbb{F} -stopping times taking values in $[t, T]$. In case L is not càdlàg, the problem of right formulation of the minimal condition is more complicated. Of course, the minimal condition must ensure uniqueness of solutions under reasonable assumptions on f . On the other hand, we want (1.4) to be satisfied. In the present paper, for optional barrier L , we propose the following minimality condition for K :

$$\int_0^T (Y_{s-} - \limsup_{u \uparrow s} L_u) dK_s^* + \sum_{s < T} (Y_s - L_s) \Delta^+ K_s = 0, \quad (1.5)$$

where K^* is the càdlàg part of process K and $\Delta^+ K_t = K_{t+} - K_t$ (i.e. $\Delta^+ K_t$ is the right-side jump of K). Under this condition (Y, Z) satisfies (1.4). Note that if L and K are càdlàg, then (1.5) reduces to (1.3).

The fundamental results on RBSDEs with Brownian filtration, one continuous barrier and L^2 -data were obtained in [6]. These results were generalized to equations with two continuous barriers in [2, 9]. Equations with continuous barriers and L^p -data with $p \in [1, 2)$ were studied for instance in [4, 11, 13, 23]. In most papers devoted to RBSDEs with possibly discontinuous barriers it is assumed that the barriers are càdlàg (see, e.g., [10, 19, 20] and the references therein). In [22] (the case $p = 2$) and in [12] (the case $p \in [1, 2]$) progressively measurable barriers are considered. In these papers the minimality condition for K differs from (1.3) and from (1.5), and what is more important here, the first component Y of the solution of (1.1) need not satisfy (1.2), but satisfies weaker condition saying that $Y_t \geq L_t$ for a.e. $t \in [0, T]$. A serious drawback to the last condition is that it does not lead to (1.4). In fact, in case $f = 0$ and $V = 0$, the first component Y of the solution of (1.1) defined in [12, 22] is the strong envelope of L (for the notion of strong envelope see [24]). It is worth noting, however, that the definition of a solution of (1.1) adopted in [12, 22] is suitable for applications to the obstacle problem for parabolic PDEs (see [14]).

To our knowledge, the paper by Grigorova et al. [8] is the only paper dealing with RBSDEs with barriers that are not càdlàg, and whose solution satisfies (1.2) and (1.4). In the present paper we prove existence and uniqueness results for (1.1) which generalize the corresponding results of [8] in several directions. First of all, we impose no regularity assumptions on L (in [8] it is assumed that L is left-limited and right upper-semicontinuous). Secondly, we consider the case of L^p -data with $p \geq 1$ (in [8] only the case of $p = 2$ is considered). As for the generator, we assume that it is Lipschitz continuous with respect to z and only continuous and monotone with respect to y (in [8] it is assumed that f is Lipschitz continuous with respect to y and z). Let us also

stress that the proofs of our results are totally different from those of [8]. Our main new idea is to reduce the problem for optional barriers to the problem for càdlàg barriers.

In Section 4 we consider the problem of approximation of solutions of (1.1) by solutions of usual BSDEs (this problem was not considered in [8]). We show that the solution of (1.1) is the increasing limit of the sequence $\{Y^n\}$ of solutions of the following penalized BSDEs

$$\begin{aligned} Y_t^n = & \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T dV_s - \int_t^T Z_s^n dB_s \\ & + n \int_t^T (Y_s^n - L_s)^- ds + \sum_{t \leq \sigma_{n,i} < T} (Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-, \quad t \in [0, T] \end{aligned}$$

with specially defined arrays of stopping times $\{\{\sigma_{n,i}\}\}$ exhausting right-side jumps of L and V . If L, V are càdlàg then the term involving the right-side jumps vanishes and our penalization scheme reduces to the usual penalization for BSDEs with càdlàg trajectories.

2 Preliminaries

Recall that a function $y : [0, T] \rightarrow \mathbb{R}^d$ is called regulated if for every $t \in [0, T)$ the limit $y_{t+} = \lim_{u \downarrow t} y_u$ exists, and for every $s \in (0, T]$ the limit $y_{s-} = \lim_{u \uparrow s} y_u$ exists. For any regulated function y on $[0, T]$ we set $\Delta^+ y_t = y_{t+} - y_t$ if $0 \leq t < T$, and $\Delta^- y_s = y_s - y_{s-}$ if $0 < s \leq T$ with the convention that $\Delta^+ y_T = \Delta^- y_0 = 0$ and $\Delta y_t = \Delta^+ y_t + \Delta^- y_t$, $t \in [0, T]$. It is known that each regulated function is bounded and has at most countably many discontinuities (see, e.g., [3, Chapter 2, Corollary 2.2]).

For $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{d \times n}$ we set $|x|^2 = \sum_{i=1}^d |x_i|^2$, $\|z\|^2 = \text{trace}(z^* z)$. $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in \mathbb{R}^d and $\text{sgn}(x) = \mathbf{1}_{\{x \neq 0\}} x/|x|$.

By L^p , $p > 0$, we denote the space of random variables X such that $\|X\|_p \equiv E(|X|^p)^{1/p} < \infty$. By \mathcal{S} we denote the set of all \mathbb{F} -progressively measurable processes with regulated trajectories, and by \mathcal{S}^p , $p > 0$, the subset of $Y \in \mathcal{S}$ such that $E \sup_{0 \leq t \leq T} |Y_t|^p < \infty$. \mathcal{H} is the set of d -dimensional \mathbb{F} -progressively measurable processes X such that

$$P\left(\int_0^T |X_t|^2 dt < \infty\right) = 1,$$

and \mathcal{H}^p , $p > 0$, is the set of all $X \in \mathcal{H}$ such that $\|X\|_{\mathcal{H}^p} \equiv \|(\int_0^T |X_s|^2 ds)^{1/2}\|_p < +\infty$.

We say that an \mathbb{F} -progressively measurable process X is of class (D) if the family $\{X_\tau, \tau \in \Gamma\}$ is uniformly integrable, where Γ is the set of all \mathbb{F} -stopping times taking values in $[0, T]$. We equip the space of processes of class (D) with the norm $\|X\|_D = \sup_{\tau \in \Gamma} E|X_\tau|$.

For $\tau \in \Gamma$, by $[[\tau]]$ we denote the set $\{(\omega, t) : \tau(\omega) = t\}$. A sequence $\{\tau_k\} \subset \Gamma$ is called stationary if

$$\forall \omega \in \Omega \quad \exists n \in \mathbb{N} \quad \forall k \geq n \quad \tau_k(\omega) = T.$$

\mathcal{M}_{loc} (resp. \mathcal{M}) is the set of all \mathbb{F} -martingales (resp. local martingales) M such that $M_0 = 0$. \mathcal{M}^p , $p \geq 1$, denotes the space of all $M \in \mathcal{M}$ such that

$$E([M]_T)^{p/2} < \infty,$$

where $[M]$ stands for the quadratic variation of M .

\mathcal{V} (resp. \mathcal{V}^+) denotes the space of \mathbb{F} -progressively measurable process of finite variation (resp. increasing) such that $V_0 = 0$, and \mathcal{V}^p (resp. $\mathcal{V}^{+,p}$), $p \geq 1$, is the set of processes $V \in \mathcal{V}$ (resp. $V \in \mathcal{V}^+$) such that $E|V|_T^p < \infty$, where $|V|_T$ denotes the total variation of V on $[0, T]$. For $V \in \mathcal{V}$, by V^* we denote the càdlàg part of the process V , and by V^d its purely jumping part consisting of right jumps, i.e.

$$V_t^d = \sum_{s < t} \Delta^+ V_s, \quad V_t^* = V_t - V_t^d, \quad t \in [0, T].$$

Let $V^1, V^2 \in \mathcal{V}$. We write $dV^1 \leq dV^2$ if $dV^{1,*} \leq dV^{2,*}$ and $\Delta^+ V^1 \leq \Delta^+ V^2$ on $[0, T]$.

In the whole paper all relations between random variables hold P -a.s. For process X, Y we write $X \leq Y$ if $X_t \leq Y_t$, $t \in [0, T]$. For a given optional process L of class (D) we set

$$\text{Snell}(L)_t = \text{ess sup}_{\tau \in \Gamma_t} E(L_\tau | \mathcal{F}_t),$$

where Γ_t is the set of all stopping times taking values in $[t, T]$. From [5] it follows that the process $\text{Snell}(L)$ is the smallest supermartingale dominating the process L .

We will need the following assumptions.

(H1) There is $\lambda \geq 0$ such that $|f(t, y, z) - f(t, y, z')| \leq \lambda|z - z'|$ for all $t \in [0, T]$, $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

(H2) there is $\mu \in \mathbb{R}$ such that $(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu(y - y')^2$ for all $t \in [0, T]$, $y, y' \in \mathbb{R}$, $z \in \mathbb{R}^d$.

(H3) $\xi, \int_0^T |f(r, 0, 0)| dr, |V|_T \in L^p$,

(H4) for every $(t, z) \in [0, T] \times \mathbb{R}^d$ the mapping $\mathbb{R} \ni y \rightarrow f(t, y, z)$ is continuous,

(H5) $[0, T] \ni t \mapsto f(t, y, 0) \in L^1(0, T)$ for every $y \in \mathbb{R}$,

(H6) there exists a process X such that $E \sup_{0 \leq t \leq T} |X_t|^p < \infty$, $X \in \mathcal{M}_{loc} + \mathcal{V}^p$, $X \geq L$ and $\int_0^T f^-(s, X_s, 0) ds \in L^p$,

(H6*) there exists a process X of class (D) such that $X \in \mathcal{M}_{loc} + \mathcal{V}^1$, $X \geq L$ and $\int_0^T f^-(s, X_s, 0) ds \in L^1$,

(Z) there exists a progressively measurable process g and $\gamma \geq 0$, $\alpha \in [0, 1)$ such that

$$|f(t, y, z) - f(t, y, 0)| \leq \gamma(g_t + |y| + |z|)^\alpha, \quad t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d.$$

Definition 2.1. We say that a pair (Y, Z) of \mathbb{F} -progressively measurable processes is a solution of BSDE with right-hand side $f + dV$ and terminal condition ξ (BSDE($\xi, f + dV$) in short) if

(a) $(Y, Z) \in \mathcal{S}^p \times \mathcal{H}$ for some $p > 1$ or Y is of class (D) and $Z \in \mathcal{H}^q$ for $q \in (0, 1)$,

(b) $\int_0^T |f(s, Y_s, Z_s)| ds < \infty$,

(c) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dV_s - \int_t^T Z_s dB_s$, $t \in [0, T]$.

Theorems 2.2 and 2.3 below were proved in [12, Section 4] in case V is càdlàg. In the general case, i.e. if $V \in \mathcal{V}$, their proofs go without any changes. The only difference is that we use Itô's formula for regulated processes (see Appendix) instead the usual Itô's formula.

Theorem 2.2. *Let $p > 1$. If (H1)–(H5) are satisfied then there exists a unique solution (Y, Z) of BSDE($\xi, f + dV$). Moreover, $Z \in \mathcal{H}^p$ and $E(\int_0^T |f(s, Y_s, Z_s)| ds)^p < \infty$.*

Theorem 2.3. *Let $p = 1$. If (H1)–(H5), (Z) are satisfied then there exists a unique solution (Y, Z) of BSDE($\xi, f + dV$). Moreover, $Y \in \mathcal{S}^q$ for every $q \in (0, 1)$ and $E \int_0^T |f(s, Y_s, Z_s)| ds < \infty$.*

Now we recall the definition of a solution of the reflected BSDE in the class of càdlàg processes and results about existence and uniqueness. Theorems 2.5 and 2.6 below were proved in [12].

Definition 2.4. Assume that L, V are càdlàg processes. We say that a triple (Y, Z, K) of \mathbb{F} -progressively measurable processes is a solution of reflected BSDE with right-hand side $f + dV$, terminal condition ξ and lower barrier L (RBSDE($\xi, f + dV, L$) in short) if

- (a) $(Y, Z) \in \mathcal{S}^p \times \mathcal{H}$ for some $p > 1$ or Y is of class (D) and $Z \in \mathcal{H}^q$ for $q \in (0, 1)$,
- (b) $K \in \mathcal{V}^+$ is càdlàg, $Y_t \geq L_t$, $t \in [0, T]$, and $\int_0^T (Y_{s-} - L_{s-}) dK_s = 0$,
- (c) $\int_0^T |f(s, Y_s, Z_s)| ds < \infty$,
- (d) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s dB_s$, $t \in [0, T]$.

Theorem 2.5. *Let $p > 1$ and (H1)–(H6) be satisfied. Then there exists a unique solution (Y, Z, K) of RBSDE($\xi, f + dV, L$). Moreover, $(Y, Z, K) \in \mathcal{S}^p \otimes \mathcal{H}^p \otimes \mathcal{V}^{+,p}$ and $E(\int_0^T |f(s, Y_s, Z_s)| ds)^p < \infty$.*

Theorem 2.6. *Let $p = 1$ and (H1)–(H5), (H6*), (Z) be satisfied. Then there exists a unique solution (Y, Z, K) of RBSDE(ξ, f, L). Moreover, Y is of class (D), $(Y, Z, K) \in \mathcal{S}^q \otimes \mathcal{H}^q \otimes \mathcal{V}^{1,+}$ for $q \in (0, 1)$ and $E \int_0^T |f(s, Y_s, Z_s)| ds < \infty$.*

For convenience of the reader we now formulate counterparts of [12, Lemma 4.11] and [12, Theorem 4.12] for regulated processes.

Lemma 2.7. *Assume that (H1)–(H4) hold. Let $L^n, L \in \mathcal{V}$, g_n, g, \bar{f} be progressively measurable processes such that $\int_0^T |g_n(s)| ds, \int_0^T |g(s)| ds, \int_0^T |\bar{f}(s)| ds \in L^1$, and let $(Y^n, Z^n), (Y, Z) \in \mathcal{S} \otimes \mathcal{H}$ be such that $t \mapsto f(t, Y_t^n, Z_t^n), t \mapsto f(t, Y_t, Z_t) \in L^1(0, T)$ and*

$$Y_t^n = Y_0^n - \int_0^t g_n(s) ds - \int_0^t f(s, Y_s^n, Z_s^n) ds - \int_0^t dL_s^n + \int_0^t Z_s^n dB_s, \quad t \in [0, T],$$

$$Y_t = Y_0 - \int_0^t g(s) ds - \int_0^t \bar{f}(s) ds - \int_0^t dL_s + \int_0^t Z_s dB_s, \quad t \in [0, T].$$

If

- (a) $E \sup_{n \geq 0} (L^n)_T^+ + E \int_0^T |f(s, 0, 0)| ds < \infty$,

(b) $\liminf_{n \rightarrow \infty} (\int_{\sigma}^{\tau} (Y_s - Y_s^n) dL_s^{n,*} + \sum_{\sigma \leq s < \tau} (Y_s - Y_s^n) \Delta^+ L_s^n) \geq 0$ for all $\sigma, \tau \in \Gamma$ such that $\sigma \leq \tau$,

(c) there exists $C \in \mathcal{V}^{1,+}$ such that $|\Delta^-(Y_t - Y_t^n)| \leq |\Delta^- C_t|$, $t \in [0, T]$,

(d) there exist processes $\underline{y}, \overline{y} \in \mathcal{V}^{1,+} + \mathcal{M}_{loc}$ of class (D) such that

$$\overline{y}_t \leq Y_t \leq \underline{y}_t, \quad t \in [0, T], \quad E \int_0^T f^+(s, \overline{y}_s, 0) ds + E \int_0^T f^-(s, \underline{y}_s, 0) ds < \infty,$$

(e) there exists $h \in L^1(\mathcal{F})$ such that $|g_n(s)| \leq h(s)$ for a.e. $s \in [0, T]$,

(f) $Y_t^n \rightarrow Y_t$, $t \in [0, T]$,

then

$$Z^n \rightarrow Z, \quad \lambda \otimes P\text{-a.e.}, \quad \int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds \rightarrow 0 \quad \text{in probability } P$$

and there exists a sequence $\{\tau_k\} \subset \Gamma$ such that for all $k \in \mathbb{N}$ and $p \in (0, 2)$,

$$E \int_0^{\tau_k} |Z_s^n - Z_s|^p ds \rightarrow 0. \quad (2.1)$$

If $\Delta^- C_t = 0$, $t \in [0, T]$, then (2.1) also holds for $p = 2$. If additionally $g_n \rightarrow g$ weakly in $L^1([0, T] \times \Omega)$ and $L_{\tau}^n \rightarrow L_{\tau}$ weakly in L^1 for every $\tau \in \Gamma$, then $\bar{f}(s) = f(s, Y_s, Z_s)$ for a.e. $s \in [0, T]$.

Proof. It is enough to repeat step by step the the proof [12, Lemma 4.11] and use Itô's formula for regulated processes (see Appendix). The only difference is that inequality (4.16) in [12] in our case takes the form

$$\begin{aligned} E \int_{\sigma}^{\tau} |Z_s - Z_s^n|^2 ds &\leq E |Y_{\tau} - Y_{\tau}^n|^2 + 2E \int_{\sigma}^{\tau} |Y_s - Y_s^n| |f(s, Y_s, Z_s) - f(s, Y_s^n, Z_s^n)| ds \\ &\quad + 2 \int_{\sigma}^{\tau} |Y_s - Y_s^n| |g(s) - g_n(s)| ds + 2E \int_{\sigma}^{\tau} (Y_s - Y_s^n) d(L_s - L_s^n)^* \\ &\quad + 2E \sum_{\sigma \leq s < \tau} (Y_s - Y_s^n) \Delta^+(L_s - L_s^n) + E \sum_{\sigma \leq s < \tau} |\Delta^-(L_s - L_s^n)|^2. \end{aligned}$$

□

Remark 2.8. In Lemma 2.7 assumption (e) may be replaced by the following one: there exists a stationary sequence $\{\tau_k\} \subset \Gamma$ such that $\sup_{n \geq 1} E \int_0^{\tau_k} |g_n(s)|^2 ds < \infty$ and the assertion of the lemma holds. This follows from the fact that assumption (e) is used in the proof of [12, Lemma 4.11] only to show that [12, (4.15)] holds true, i.e. that $\int_0^T |g(s) - g_n(s)| |Y_s - Y_s^n| ds \rightarrow 0$. But under the new condition this follows from the inequality

$$\int_0^T |g(s) - g_n(s)| |Y_s - Y_s^n| ds \leq \left(E \int_0^T |g(s) - g_n(s)|^2 ds \right)^{1/2} \left(E \int_0^T |Y_s - Y_s^n|^2 ds \right)^{1/2}.$$

Theorem 2.9. Assume that (H1)–(H4) hold, $(Y^n, Z^n) \in \mathcal{S} \otimes \mathcal{H}$, $A^n \in \mathcal{V}$, $K^n \in \mathcal{V}^+$, $t \mapsto f(t, Y_t^n, Z_t^n) \in L^1(0, T)$ and

$$Y_t^n = Y_0^n - \int_0^t g_n(s) ds - \int_0^t f(s, Y_s^n, Z_s^n) ds - \int_0^t dK_s^n + \int_0^t dA_s^n + \int_0^t Z_s^n dB_s$$

for $t \in [0, T]$. Moreover, assume that

- (a) $dA^n \leq dA^{n+1}$, $n \in \mathbb{N}$, $\sup_{n \geq 0} E|A^n|_T < \infty$,
- (b) $\liminf_{n \rightarrow \infty} \left(\int_\sigma^\tau (Y_s - Y_s^n) d(K_s^n - A_s^n)^* + \sum_{\sigma \leq s < \tau} (Y_s - Y_s^n) \Delta^+(K_s^n - A_s^n) \right) \geq 0$ for any $\sigma, \tau \in \Gamma$ such that $\sigma \leq \tau$,
- (c) there exists process $C \in \mathcal{V}^{1,+}$ such that $\Delta^- K_t^n \leq \Delta^- C_t$, $t \in [0, T]$,
- (d) there exist processes $\underline{y}, \bar{y} \in \mathcal{V}^{1,+} + \mathcal{M}_{loc}$ of class (D) such that
$$E \int_0^T f^+(s, \bar{y}_s, 0) ds + E \int_0^T f^-(s, \underline{y}_s, 0) ds < \infty, \quad \bar{y}_t \leq Y_t^n \leq \underline{y}_t, \quad t \in [0, T],$$
- (e) $E \int_0^T |f(s, 0, 0)| ds < \infty$ and there exists a progressively measurable process $h \in L^1([0, T] \times \Omega)$ such that $|g_n(s)| \leq h(s)$ for a.e. $s \in [0, T]$,
- (f) $Y_t^n \nearrow Y_t$, $t \in [0, T]$.

Then $Y \in \mathcal{S}$ and there exist $K \in \mathcal{V}^+$, $A \in \mathcal{V}^1$, $Z \in \mathcal{H}$ and progressively measurable process $g \in L^1([0, T] \times \Omega)$ such that

$$Y_t = Y_0 - \int_0^t g(s) ds - \int_0^t f(s, Y_s, Z_s) ds - \int_0^t dK_s + \int_0^t dA_s + \int_0^t Z_s dB_s \quad t \in [0, T]$$

and

$$Z^n \rightarrow Z, \quad \lambda \otimes P\text{-a.e.}, \quad \int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds \rightarrow 0 \quad \text{in probability } P.$$

Moreover, there exists a stationary sequence $\{\tau_k\} \subset \Gamma$ such that for every $p \in (0, 2)$,

$$E \int_0^{\tau_k} |Z_s^n - Z_s|^p ds \rightarrow 0. \tag{2.2}$$

If $|\Delta^- C_t| + |\Delta^- K_t| = 0$, $t \in [0, T]$, then (2.2) also holds for $p = 2$.

Remark 2.10. Since the proof of the above theorem follows directly from Lemma 2.7, it suffices to assume in (e) that there exists a stationary sequence $\{\tau_k\}$ such that $\sup_{n \geq 1} E \int_0^{\tau_k} |g_n(s)|^2 ds < \infty$ (see Remark 2.8).

3 Reflected BSDEs

In what follows we assume that the barrier L is an \mathbb{F} -adapted optional process and that $\xi \geq L_T$.

Definition 3.1. We say that a triple (Y, Z, K) of \mathbb{F} -progressively measurable processes is a solution of the reflected backward stochastic differential equation with right-hand side $f + dV$, terminal value ξ and lower barrier L (RBSDE($\xi, f + dV, L$)) if

- (a) $(Y, Z) \in \mathcal{S}^p \otimes \mathcal{H}$ for some $p > 1$ or Y is of class (D) and $Z \in \mathcal{H}^q$ for $q \in (0, 1)$,
- (b) $K \in \mathcal{V}^+$, $L_t \leq Y_t$, $t \in [0, T]$, and

$$\int_0^T (Y_{s-} - \limsup_{u \uparrow s} L_u) dK_s^* + \sum_{s < T} (Y_s - L_s) \Delta^+ K_s = 0,$$

$$(c) \int_0^T |f(s, Y_s, Z_s)| ds < \infty,$$

$$(d) Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dK_s + \int_t^T dV_s - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

Remark 3.2. Assume that (Y, Z, K) is a solution of RBSDE($\xi, f + dV, L$). Let $a \in \mathbb{R}$, and let

$$\tilde{\xi} = e^{aT} \xi, \quad \tilde{L}_t = e^{at} L_t, \quad \tilde{V}_t = \int_0^t e^{as} dV_s^* + \sum_{s < t} e^{as} \Delta V_s^+,$$

$$\tilde{f}(t, y, z) = e^{at} f(t, e^{-at} y, e^{-at} z) - ay$$

and

$$\tilde{Y}_t = e^{at} Y_t, \quad \tilde{Z}_t = e^{at} Z_t, \quad \tilde{K}_t = \int_0^t e^{as} dK_s^* + \sum_{s < t} e^{as} \Delta K_s^+.$$

Then $(\tilde{Y}, \tilde{Z}, \tilde{K})$ solves RBSDE($\tilde{\xi}, \tilde{f} + d\tilde{V}, \tilde{L}$). Therefore choosing a appropriately we may assume that (H2) is satisfied with arbitrary but fixed $\mu \in \mathbb{R}$.

Let $\{\sigma_k^i\}$ be a finite sequence of stopping times and let (Y^i, Z^i, A^i) be a solution of the following BSDE

$$\begin{aligned} Y_t^i &= \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i) ds + \int_t^T dV_s^i - \int_t^T Z_s^i dB_s \\ &\quad + \sum_{t \leq \sigma_k^i < T} (Y_{\sigma_k^i}^i + \Delta^+ V_{\sigma_k^i}^i - L_{\sigma_k^i}^i)^-, \quad t \in [0, T], \quad i = 1, 2. \end{aligned} \quad (3.1)$$

Proposition 3.3. Assume that f^1 satisfies (H1) and (H2), $\xi^1 \leq \xi^2$, $f^1(t, Y_t^2, Z_t^2) \leq f^2(t, Y_t^2, Z_t^2)$, $dV_t^1 \leq dV_t^2$, $U_t^1 \leq U_t^2$, $t \in [0, T]$, and $\bigcup_k [[\sigma_k^1]] \subset \bigcup_k [[\sigma_k^2]]$. If $(Y^1 - Y^2)^+ \in \mathcal{S}^p$ for some $p > 1$, then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$.

Proof. Let $q > 1$ be such that $(Y^1 - Y^2)^+ \in \mathcal{S}^q$ and $p \in (1, q)$. Without loss of generality we may assume that $\mu = -\frac{4\lambda}{p-1}$. By (H1) and (H2),

$$\begin{aligned} &((Y_s^1 - Y_s^2)^+)^{p-1} (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) \\ &\leq ((Y_s^1 - Y_s^2)^+)^{p-1} (f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)) \\ &\leq -\frac{4\lambda}{p-1} ((Y_s^1 - Y_s^2)^+)^p + \lambda ((Y_s^1 - Y_s^2)^+)^{p-1} |Z_s^1 - Z_s^2| \end{aligned}$$

for $s \in [0, T]$. By Corollary 5.5, for $\tau, \sigma \in \Gamma$ such that $\tau \leq \sigma$ we have

$$\begin{aligned}
& ((Y_\tau^1 - Y_\tau^2)^+)^p + \frac{p(p-1)}{2} \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-2} \mathbf{1}_{\{Y_s^1 > Y_s^2\}} |Z_s^1 - Z_s^2|^2 ds \\
& \leq ((Y_\sigma^1 - Y_\sigma^2)^+)^p + p \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-1} (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) ds \\
& \quad + p \int_\tau^\sigma ((Y_{s-}^1 - Y_{s-}^2)^+)^{p-1} d(V_s^1 - V_s^2)^* + p \sum_{\tau \leq s < \sigma} ((Y_s^1 - Y_s^2)^+)^{p-1} \Delta^+(V_s^1 - V_s^2) \\
& \quad + p \sum_{\tau \leq \sigma_k^1 < \sigma} ((Y_{\sigma_k^1}^1 - Y_{\sigma_k^1}^2)^+)^{p-1} (Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 - L_{\sigma_k^1})^- \\
& \quad - p \sum_{\tau \leq \sigma_k^2 < \sigma} ((Y_{\sigma_k^2}^1 - Y_{\sigma_k^2}^2)^+)^{p-1} (Y_{\sigma_k^2+}^2 + \Delta^+ V_{\sigma_k^2}^2 - L_{\sigma_k^2})^- \\
& \quad - p \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-1} (Z_s^1 - Z_s^2) dB_s.
\end{aligned}$$

By the above and the assumptions,

$$\begin{aligned}
& ((Y_\tau^1 - Y_\tau^2)^+)^p + \frac{p(p-1)}{2} \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-2} \mathbf{1}_{\{Y_s^1 > Y_s^2\}} |Z_s^1 - Z_s^2|^2 ds \\
& \leq ((Y_\sigma^1 - Y_\sigma^2)^+)^p - \frac{4\lambda}{p-1} \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^p ds + \lambda \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-1} |Z_s^1 - Z_s^2| ds \\
& \quad + p \sum_{\tau \leq \sigma_k^1 < \sigma} ((Y_{\sigma_k^1}^1 - Y_{\sigma_k^1}^2)^+)^{p-1} (Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 - L_{\sigma_k^1})^- \\
& \quad - p \sum_{\tau \leq \sigma_k^2 < \sigma} ((Y_{\sigma_k^2}^1 - Y_{\sigma_k^2}^2)^+)^{p-1} (Y_{\sigma_k^2+}^2 + \Delta^+ V_{\sigma_k^2}^2 - L_{\sigma_k^2})^- \\
& \quad - p \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-1} (Z_s^1 - Z_s^2) dB_s. \tag{3.2}
\end{aligned}$$

Since $\bigcup_k [[\sigma_k^1]] \subset \bigcup_k [[\sigma_k^2]]$,

$$\begin{aligned}
& \sum_{\tau \leq \sigma_k^1 < \sigma} ((Y_{\sigma_k^1}^1 - Y_{\sigma_k^1}^2)^+)^{p-1} (Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 - L_{\sigma_k^1})^- \\
& \quad - \sum_{\tau \leq \sigma_k^2 < \sigma} ((Y_{\sigma_k^2}^1 - Y_{\sigma_k^2}^2)^+)^{p-1} (Y_{\sigma_k^2+}^2 + \Delta^+ V_{\sigma_k^2}^2 - L_{\sigma_k^2})^- \\
& \leq \sum_{\tau \leq \sigma_k^1 < \sigma} ((Y_{\sigma_k^1}^1 - Y_{\sigma_k^1}^2)^+)^{p-1} \{ (Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 - L_{\sigma_k^1})^- - (Y_{\sigma_k^1+}^2 + \Delta^+ V_{\sigma_k^1}^2 - L_{\sigma_k^1})^- \} =: I.
\end{aligned}$$

We shall show that $I \leq 0$. Under the assumption that $Y_{\sigma_k^1}^1 \leq Y_{\sigma_k^1}^2$ the this inequality is obvious. Assume now that $Y_{\sigma_k^1}^1 > Y_{\sigma_k^1}^2$. By (4.3),

$$Y_{\sigma_k^i}^i = (Y_{\sigma_k^i+}^i + \Delta^+ V_{\sigma_k^i}^i) \vee L_{\sigma_k^i}, \quad i = 1, 2. \tag{3.3}$$

We have $Y_{\sigma_k^1}^1 > Y_{\sigma_k^1}^2 \geq L_{\sigma_k^1}$. By this and (3.3), $Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 \geq L_{\sigma_k^1}$. Hence $(Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 - L_{\sigma_k^1})^- = 0$, which implies that

$$I = - \sum_{\tau \leq \sigma_k^1 < T} ((Y_{\sigma_k^1}^1 - Y_{\sigma_k^1}^2)^+)^{p-1} (Y_{\sigma_k^1+}^2 + \Delta^+ V_{\sigma_k^1}^2 - L_{\sigma_k^1})^- \leq 0. \tag{3.4}$$

Note that

$$\begin{aligned}
& p\lambda((Y_s^1 - Y_s^2)^+)^{p-1}|Z_s^1 - Z_s^2| \\
&= p((Y_s^1 - Y_s^2)^+)^{p-2}\mathbf{1}_{\{Y_s^1 > Y_s^2\}}(\lambda(Y_s^1 - Y_s^2)^+|Z_s^1 - Z_s^2|) \\
&\leq p((Y_s^1 - Y_s^2)^+)^{p-2}\mathbf{1}_{\{Y_s^1 > Y_s^2\}}\left(\frac{4\lambda^2}{p-1}((Y_s^1 - Y_s^2)^+)^2 + \frac{p-1}{4}|Z_s^1 - Z_s^2|^2\right) \\
&= \frac{4p\lambda^2}{p-1}((Y_s^1 - Y_s^2)^+)^p + \frac{p(p-1)}{4}((Y_s^1 - Y_s^2)^+)^{p-2}\mathbf{1}_{\{Y_s^1 > Y_s^2\}}|Z_s^1 - Z_s^2|^2
\end{aligned}$$

for $s \in [0, T]$. From this and (3.2), (3.4) it follows that

$$\begin{aligned}
& ((Y_\tau^1 - Y_\tau^2)^+)^p + \frac{p(p-1)}{4} \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-2}\mathbf{1}_{\{Y_s^1 > Y_s^2\}}|Z_s^1 - Z_s^2|^2 ds \\
&\leq ((Y_\sigma^1 - Y_\sigma^2)^+)^p - p \int_\tau^\sigma ((Y_s^1 - Y_s^2)^+)^{p-1}(Z_s^1 - Z_s^2) dB_s \\
&= ((Y_\sigma^1 - Y_\sigma^2)^+)^p + M_\sigma - M_\tau.
\end{aligned} \tag{3.5}$$

Let $\{\sigma_k\} \in \Gamma$ be a fundamental sequence for the local martingale M . Changing σ_k with σ in the above inequality, taking expected value and passing to the limit with $k \rightarrow \infty$ we get $E(Y_\tau^1 - Y_\tau^2)^+ = 0$. By The Section Theorem, $(Y_t^1 - Y_t^2)^+ = 0$, $t \in [0, T]$. \square

Remark 3.4. Observe that if f, f' do not depend on z then it is enough to assume that $(Y - Y')^+$ is of class (D).

Remark 3.5. Let $f^1, f^2, \xi^1, \xi^2, dV^1, dV^2, \bigcup_k [[\sigma_k^1]], \bigcup_k [[\sigma_k^2]]$ be satisfying the same assumptions as in Proposition 3.3. Moreover, assume that f^1 satisfies (Z) and $Z^1, Z^2 \in L^q((0, T) \otimes \Omega)$ for some $q \in (\alpha, 1]$. Then $(Y^1 - Y^2)^+ \in \mathcal{S}^p$ for some $p > 1$.

Proof. By Corollary (5.5), assumptions on the data and (3.4),

$$\begin{aligned}
(Y_t^1 - Y_t^2)^+ &\leq (\xi^1 - \xi^2)^+ + \int_t^T \mathbf{1}_{\{Y_s^1 > Y_s^2\}}(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) ds \\
&\quad + \int_t^T \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} d(V_s^1 - V_s^2)^* + \sum_{t \leq s < T} \mathbf{1}_{\{Y_s^1 > Y_s^2\}} \Delta^+(V_s^1 - V_s^2) \\
&\quad - \int_t^T \mathbf{1}_{\{Y_s^1 > Y_s^2\}}(Z_s^1 - Z_s^2) dB_s \\
&\quad + \sum_{t \leq \sigma_k^1 < T} \mathbf{1}_{\{Y_s^1 > Y_s^2\}}(Y_{\sigma_k^1+}^1 + \Delta^+ V_{\sigma_k^1}^1 - L_{\sigma_k^1}^1)^- \\
&\quad - \sum_{t \leq \sigma_k^2 < T} \mathbf{1}_{\{Y_s^1 > Y_s^2\}}(Y_{\sigma_k^2+}^2 + \Delta^+ V_{\sigma_k^2}^2 - L_{\sigma_k^2}^2)^- \\
&\leq \int_t^T \mathbf{1}_{\{Y_s^1 > Y_s^2\}}(f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)) ds \\
&\quad - \int_t^T \mathbf{1}_{\{Y_s^1 > Y_s^2\}}(Z_s^1 - Z_s^2) dB_s.
\end{aligned}$$

Note that by (Z),

$$\begin{aligned} |f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)| &\leq |f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, 0)| \\ &\quad + |f^1(s, Y_s^2, 0) - f^1(s, Y_s^2, Z_s^2)| \leq 2\gamma(g_s + |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2|)^\alpha \end{aligned}$$

for $s \in [0, T]$. Hence

$$(Y_t^1 - Y_t^2)^+ \leq 2\gamma E\left(\int_0^T (g_s + |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2|)^\alpha ds \middle| \mathcal{F}_t\right).$$

Let $p > 1$ be such that $\alpha \cdot p = q$. By Doob's inequality,

$$E \sup_{t \leq T} ((Y_t^1 - Y_t^2)^+)^p \leq C_p E\left(\int_0^T (g_s + |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2|)^q ds\right).$$

Hence $(Y^1 - Y^2)^+ \in \mathcal{S}^p$. □

Lemma 3.6. *Let $x : [0, T] \rightarrow \mathbb{R}$ be nonnegative, and measurable and $y : [0, T] \rightarrow \mathbb{R}$ be nondecreasing and continuous. If for every $t \in (0, T]$ such that $x(t) > 0$ there exists $\varepsilon_t > 0$ such that $\int_{t-\varepsilon_t}^t x(s) dy(s) = 0$, then $\int_0^T x(s) dy(s) = 0$.*

Proof. Suppose that $\int_0^T x(s) dy(s) > 0$. Set $F(t) = \int_0^t x(s) dy(s)$, $t \in [0, T]$. It is well known that the function

$$f(t) \equiv \liminf_{\varepsilon \searrow 0} \frac{F(t) - F(t - \varepsilon)}{y(t) - y(t - \varepsilon)} \quad (3.6)$$

is Borel measurable and $f = x$, dy -a.e. Let

$$A = \{t \in (0, T] : x(t) > 0\}, \quad B = \{t \in (0, T] : f(t) = x(t)\}.$$

By the assumption, $dy(A \cap B) > 0$. Let $t \in A \cap B$. Then $x(t) > 0$ and by (3.6), $\int_{t-\varepsilon}^t x(s) dy(s) > 0$ for every $\varepsilon > 0$, which contradicts the assumption. □

Proposition 3.7. *Assume that Y is the Snell envelope of the optional process L of class (D). Let A be the continuous part of the increasing process K from Mertens decomposition of Y . Then*

$$\int_0^T (Y_r - \underline{L}_r) dA_r = 0, \quad (3.7)$$

where $\underline{L}_t = \limsup_{s \nearrow t} L_s$.

Proof. We may assume that L is nonnegative, otherwise since L is of class (D) there exists uniformly integrable martingale M such that $L + M$ is nonnegative. We consider then $\tilde{L} = L + M$. Its Snell envelope is equal to $\tilde{Y} = Y + M$, and obviously the finite variation part of Mertens decomposition of \tilde{Y} is equal to the finite variation part of Mertens decomposition of Y , so its continuous parts are also equal. Therefore if we prove that the assertion of the proposition holds for \tilde{L} then we would have

$$\int_0^T (Y_r - \underline{L}_r) dA_r = \int_0^T (\tilde{Y}_r - \tilde{\underline{L}}_r) dA_r = 0.$$

By [5, Proposition 2.34, p. 131], for any $t \in [0, T)$ and $\lambda > 0$,

$$\int_t^{D_t^\lambda} (Y_r - \underline{L}_r) dA_r = 0, \quad P\text{-a.s.}, \quad (3.8)$$

where $D_t^\lambda = \inf\{r \geq t, \lambda Y_r \leq L_r\} \wedge T$. Let $\Omega_{t,\lambda}$ be the set of those $\omega \in \Omega$ for which the above equality holds. Set

$$\Omega_0 = \bigcap_{t \in [0, T) \cap \mathbb{Q}, \lambda \in \mathbb{Q}^+} \Omega_{t,\lambda}.$$

It is obvious that $P(\Omega_0) = 1$. We will show that for every $\omega \in \Omega_0$ the following property holds:

$$\forall t \in (0, T] : Y_t > \underline{L}_t \quad \exists \varepsilon_t > 0 \quad \int_{t-\varepsilon_t}^t (Y_r - \underline{L}_r) dA_r = 0,$$

which when combined with Lemma 3.6 implies (3.7). Suppose that there exists $t \in (0, T]$ such that

$$Y_t > \underline{L}_t, \quad \int_{t-\varepsilon}^t (Y_r - \underline{L}_r) dA_r > 0, \quad \varepsilon > 0. \quad (3.9)$$

By the definition, $\underline{L}_t = \lim_{\delta \searrow 0} \sup_{t-\delta \leq s < t} L_s$. Therefore there exist $\varepsilon, \delta_1 > 0$ such that

$$Y_t \geq \sup_{t-\delta_1 \leq s < t} L_s + 2\varepsilon.$$

Since Y has only negative jumps, there exists $\delta_2 > 0$ such that

$$Y_r \geq \sup_{t-\delta_1 \leq s < t} L_s + \varepsilon, \quad r \in [t - \delta_2, t].$$

Let $\delta = \max\{\delta_1, \delta_2\}$ and $t_\delta = t - \delta$. Recall that $D_{t_\delta}^\lambda = \inf\{r \geq t_\delta, \lambda Y_r \leq L_r\} \wedge T$. Hence $D_{t_\delta}^\lambda \geq t$ for $\lambda = (\sup_{r \in [t_\delta, t]} L_r + \varepsilon/2) / \inf_{r \in [t_\delta, t]} Y_r$. It is clear that we can choose ε, δ so that λ, t_δ are rational. Therefore from (3.8) it follows that

$$\int_{t_\delta}^t (Y_r - \underline{L}_r) dA_r \leq \int_{t_\delta}^{D_{t_\delta}^\lambda} (Y_r - \underline{L}_r) dA_r = 0,$$

which contradicts (3.9). \square

Corollary 3.8. *Let Y be the Snell envelope of an optional process L of class (D), and let K be an increasing process from Mertens decomposition of Y . Then*

$$\int_0^T (Y_r - \underline{L}_r) dK_r^* = \sum_{t < T} (Y_t - \underline{L}_t) \Delta^+ K_t = 0.$$

Proof. By [5, Proposition 2.34, p. 131] we have

$$\sum_{t \leq T} (Y_{t-} - \underline{L}_t) \Delta^- K_t + \sum_{t < T} (Y_t - L_t) \Delta^+ K_t = 0.$$

Therefore the desired result follows from Proposition 3.7. \square

For optional processes Y, Z we set

$$f_{Y,Z}(t) = f(t, Y_t, Z_t), \quad t \in [0, T].$$

Proposition 3.9. *Let a triple (Y, Z, K) be a solution of $\text{RBSDE}(\xi, f + dV, L)$ such that $\int_0^T |f_{Y,Z}(s)| ds \in L^1$. Assume that L^+ is of class (D), $\xi \in L^1$, $V \in \mathcal{V}^1$. Then for $t \in [0, T]$,*

$$Y_t = \text{ess sup}_{\tau \in \Gamma_t} E\left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t\right),$$

where Γ_t is set of all stopping times taking values in $[t, T]$.

Proof. It follows from the definition of solution of $\text{RBSDE}(\xi, f + dV, L)$ and Corollary 3.8. \square

For a given process L of class (D) and integrable \mathcal{F}_T -measurable random variable ξ we denote by $\text{Snell}_\xi(L)$ the smallest supermartingale Z such that $Z_t \geq L_t$, $t \in [0, T]$ and $Z_T = \xi$. It is easy to see that $\text{Snell}_\xi(L) = \text{Snell}(L^\xi)$, where $L_t^\xi = \mathbf{1}_{\{t < T\}} L_t + \mathbf{1}_{\{t = T\}} \xi$. From Proposition 3.9 it follows that $\text{Snell}_\xi(L)$ is the first component of the solution of $\text{RBSDE}(\xi, 0, L)$.

Proposition 3.10. *Assume that there is a progressively measurable process g such that $E \int_0^T |g(r)| dr < \infty$ and $f(r, y, z) \geq g(r)$ for a.e. $r \in [0, T]$ and all $y \in \mathbb{R}$, $z \in \mathbb{R}^d$. Let*

$$\hat{L} = \text{Snell}_\xi(L + X) - X,$$

where (X, \tilde{Z}) is a solution of $\text{BSDE}(0, -g - dV)$. If a triple (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f + dV, \hat{L})$ with the property that $\int_0^T |f_{Y,Z}(s)| ds \in L^1$, then (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f + dV, L)$.

Proof. Let $(\bar{Y}, \bar{Z}, \bar{K})$ be a solution of $\text{RBSDE}(\xi, f_{Y,Z} + dV, L)$. Then $\bar{Y} + X$ is a supermartingale such that $\bar{Y}_T + X_T = \xi$ and $\bar{Y}_t + X_t \geq L_t + X_t$, $t \in [0, T]$. Thus $\bar{Y}_t + X_t \geq \text{Snell}_\xi(L + X)_t$, $t \in [0, T]$, and hence $\bar{Y}_t \geq \text{Snell}_\xi(L + X)_t - X_t = \hat{L}_t$, $t \in [0, T]$. Moreover,

$$\int_0^T (\bar{Y}_{t-} - \hat{L}_t) d\bar{K}_t^* + \sum_{t < T} (\bar{Y}_t - \hat{L}_t) \Delta^+ \bar{K}_t \leq \int_0^T (\bar{Y}_{t-} - L_t) d\bar{K}_t^* + \sum_{t < T} (\bar{Y}_t - L_t) \Delta^+ \bar{K}_t = 0.$$

Therefore $(\bar{Y}, \bar{Z}, \bar{K})$ is also a solution of $\text{RBSDE}(\xi, f_{Y,Z} + dV, \hat{L})$. By uniqueness (see Remark 3.5), $(\bar{Y}, \bar{Z}, \bar{K}) = (Y, Z, K)$. Therefore (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f_{Y,Z} + dV, L)$ or, equivalently, (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f + dV, L)$. \square

Lemma 3.11. *Let L be a regulated process such that $\Delta^-(L + V)_t \leq 0$ for $t \in (0, T]$, and let $(\bar{Y}, \bar{Z}, \bar{K})$ be a solution of $\text{RBSDE}(\xi, f + dV_+, L_+)$ such that $\int_0^T |f_{\bar{Y}, \bar{Z}}(s)| ds \in L^1$, where L_+ denotes a càdlàg process defined by $(L_+)_t = L_{t+}$. Then*

$$(Y_+, Z, K_+) = (\bar{Y}, \bar{Z}, \bar{K}),$$

where (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f_{\bar{Y}, \bar{Z}} + dV, L)$.

Proof. We will show that (Y_+, Z, K_+) is a solution of $\text{RBSDE}(\xi, f_{\bar{Y}, \bar{Z}} + dV_+, L_+)$. Since $Y \geq L$, then of course $Y_+ \geq L_+$. Therefore it suffices to show that

$$LS := \int_0^T ((Y_{t+})_- - (L_{t+})_-) dK_{t+} = 0.$$

We have

$$LS = \int_0^T (Y_{t-} - L_{t-}) dK_{t+} = \int_0^T (Y_t - L_t) dK_t^c + \sum_{0 < t \leq T} (Y_{t-} - L_{t-}) \Delta K_{t+}.$$

The first term on the right-hand side is equal to zero since (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f_{\bar{Y}, \bar{Z}} + dV, L)$. As for the second term, we will consider two cases. First suppose that $\Delta K_{t+} > 0$ and $\Delta^- K_t > 0$. Then $Y_{t-} = L_{t-}$ by the definition of a solution of $\text{RBSDE}(\xi, f_{\bar{Y}, \bar{Z}} + dV, L)$. Now suppose that $\Delta K_{t+} > 0$ and $\Delta^- K_t = 0$. Then $\Delta^+ K_t > 0$. Consequently, $Y_t = L_t$ by the definition of a solution of $\text{RBSDE}(\xi, f_{\bar{Y}, \bar{Z}} + dV, L)$. By the assumptions, $L_{t-} + V_{t-} \geq L_t + V_t$. Hence

$$Y_{t-} + V_{t-} \geq L_{t-} + V_{t-} \geq L_t + V_t = Y_t + V_t.$$

But $Y_{t-} + V_{t-} = Y_t + V_t$, since $\Delta^- K_t = 0$. Therefore $Y_{t-} = L_{t-}$. Thus, in both cases, $Y_{t-} = L_{t-}$. Hence $\sum_{0 < t \leq T} (Y_{t-} - L_{t-}) \Delta K_{t+} = 0$, and the proof is complete. \square

Corollary 3.12. *Let $p \geq 1$. Assume that (H1)–(H5) are satisfied and there exists a progressively measurable process g such that $\int_0^T |g(s)| ds \in \mathcal{H}^p$ and $f(r, y, z) \geq g(r)$ for a.e. $r \in [0, T]$. If $p > 1$ and $L^+ \in \mathcal{S}^p$ or $p = 1$, L^+ is of class (D) and (Z) is satisfied, then there exists a unique solution (Y, Z, K) of $\text{RBSDE}(\xi, f + dV, L)$. Moreover, $Y \in \mathcal{S}^p$, $Z \in \mathcal{H}^p$, $K \in \mathcal{S}^p$ if $p > 1$, and if $p = 1$, then Y is of class (D), $Y \in \mathcal{S}^q$, $Z \in \mathcal{H}^q$ for $q \in (0, 1)$, $K \in \mathcal{V}^+$.*

Proof. Define X, \hat{L} as in Proposition 3.10. By Theorem 2.5 and Theorem 2.6 there exists a solution of $(\bar{Y}, \bar{Z}, \bar{K})$ of $\text{RBSDE}(\xi, f + dV_+, \hat{L}_+)$. By Lemma 3.11,

$$(\bar{Y}, \bar{Z}, \bar{K}) = (Y_+, Z, K_+),$$

where (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f_{\bar{Y}, \bar{Z}} + dV, \hat{L})$. Hence (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f + dV, \hat{L})$, and by Proposition 3.10, it is a solution of $\text{RBSDE}(\xi, f + dV, L)$. Uniqueness follows from Proposition 3.3 and Remark 3.5. \square

Corollary 3.13. *Under the assumptions of Corollary 3.12,*

$$Y_t^n \nearrow Y_{t+}, \quad t \in [0, T],$$

where (Y, Z, K) is the solution of $\text{RBSDE}(\xi, f + dV, L)$ and (Y^n, Z^n) is the solution of the BSDE

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T n(Y_s^n - \hat{L}_s)^- ds + \int_t^T dV_s - \int_t^T Z_s^n dB_s, \quad t \in [0, T]$$

with \hat{L} defined in Proposition 3.10.

Lemma 3.14. *Let $p \geq 1$. Assume that (H1)–(H5) are satisfied if $p > 1$, and (H1)–(H5), (Z) are satisfied if $p = 1$. Let (Y^1, Z^1, K^1) , (Y^2, Z^2, K^2) be solutions of $\text{RBSDE}(\xi^1, f^1 + dV^1, L)$ and $\text{RBSDE}(\xi^2, f^2 + dV^2, L)$, respectively. Assume that $\xi^1 \leq \xi^2$, $f^1 \leq f^2$, $dV^1 \leq dV^2$ and there exists a progressively measurable process g such that $\int_0^T |g(s)| ds \in L^1$ and $f^1(r, y) \wedge f^2(r, y) \geq g(r)$ for a.e. $r \in [0, T]$. Then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$, and $dK^1 \geq dK^2$.*

Proof. By Remark 3.4, $Y^1 \leq Y^2$. By Lemma 3.11, Proposition 3.10 and [12], $dK_+^1 \geq dK_+^2$. Hence $dK^{1,c} \geq dK^{2,c}$. Moreover,

$$\begin{aligned}\Delta^+ K_t^1 &= (\hat{L}_t - Y_{t+}^1 - \Delta^+ V_t^1)^+ \geq (\hat{L}_t - Y_{t+}^2 - \Delta^+ V_t^2) = \Delta^+ K_t^2, \\ \Delta^- K_t^1 &= (\hat{L}_{t-} - Y_t^1 - \Delta^- V_t^1)^+ \geq (\hat{L}_{t-} - Y_t^2 - \Delta^- V_t^2) = \Delta^- K_t^2.\end{aligned}$$

□

Lemma 3.15. *Assume that $E \int_0^T |f_n(s) - f(s)| ds \rightarrow 0$, $E|\xi^n - \xi| \rightarrow 0$, $\|L - L^n\|_D \rightarrow 0$. Let*

$$Y_t^n = \text{ess sup}_{\tau \geq t} E\left(\int_t^\tau f_n(s) ds + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi_n \mathbf{1}_{\{\tau = T\}} \middle| \mathcal{F}_t\right).$$

Then $\|Y^n - Y\|_D \rightarrow 0$, where

$$Y_t = \text{ess sup}_{\tau \geq t} E\left(\int_t^\tau f(s) ds + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} \middle| \mathcal{F}_t\right).$$

Proof. For every $\sigma \in \Gamma$ we have

$$\begin{aligned}E|Y_\sigma - Y_\sigma^n| &\leq E \text{ess sup}_{\tau \geq \sigma} E\left(\left|\int_\sigma^\tau f(s) - f_n(s) ds + (L_\tau - L_\tau^n) \mathbf{1}_{\{\tau < T\}} + (\xi - \xi^n) \mathbf{1}_{\{\tau = T\}}\right| \middle| \mathcal{F}_\sigma\right) \\ &= \sup_{\tau \geq \sigma} E\left(E\left(\left|\int_\sigma^\tau f(s) - f_n(s) ds + (L_\tau - L_\tau^n) \mathbf{1}_{\{\tau < T\}} + (\xi - \xi^n) \mathbf{1}_{\{\tau = T\}}\right| \middle| \mathcal{F}_\sigma\right)\right) \\ &\leq \sup_{\tau \in \Gamma} E|L_\tau - L_\tau^n| + E \int_0^T |f(s) - f_n(s)| ds + E|\xi - \xi^n|,\end{aligned}$$

which converges to 0 as $n \rightarrow \infty$ by the assumptions of the lemma. □

Theorem 3.16. *Let $p \geq 1$. Assume that (H1)–(H6) are satisfied if $p > 1$, and if $p = 1$ then (H1)–(H5), (H6*), (Z) are satisfied. Then there exists a unique solution (Y, Z, K) of $\text{RBSDE}(\xi, f + dV, L)$. Moreover, $Y \in \mathcal{S}^p$, $Z \in \mathcal{H}^p$ and $K \in \mathcal{S}^p$ if $p > 0$, and if $p = 1$, then Y is of class (D), $Y \in \mathcal{S}^q$, $Z \in \mathcal{H}^q$ for $q \in (0, 1)$ and $K \in \mathcal{V}^+$.*

Proof. Let $f_n(t, y, z) = f(t, y, z) \vee (-n)$. By Corollary 3.12, for $n \geq 1$ there exists a solution (Y^n, Z^n, K^n) of $\text{RBSDE}(\xi, f_n + dV, L)$. By Lemma 3.14, $Y^n \geq Y^{n+1}$ and $dK^n \leq dK^{n+1}$, $n \geq 1$. By this and Proposition 3.3,

$$\bar{Y} \leq Y^n \leq Y^1, \quad n \geq 1, \quad (3.10)$$

where (\bar{Y}, \bar{Z}) is a solution of $\text{BSDE}(\xi, f + dV)$. By the above (H2) we have

$$|f_n(s, Y_s^n, 0)| \leq |f(s, Y_s^1, 0)| + |f(s, \bar{Y}_s, 0)|. \quad (3.11)$$

Let $\tau_k^1 = \inf\{t \geq 0 : \int_0^t |f(s, Y_s^1, 0)| ds + \int_0^t |f(s, \bar{Y}, 0)| ds > k\}$, and let $\{\tau_k^2\} \subset \Gamma$ be a stationary sequence of stopping times such that $Y^{1, \tau_k^2}, \bar{Y}^{\tau_k^2}, V^{\tau_k^2} \in \mathcal{S}^2$, $\int_0^{\tau_k^2} |f(s, 0, 0)| ds \in L^2$. Write $\tau_k = \tau_k^1 \wedge \tau_k^2$, $k \in \mathbb{N}$. By [12, Lemma 4.2] and the definition of $\{\tau_k\}$, for $q \leq 2$ we have

$$\begin{aligned} & E\left(\int_0^{\tau_k} |Z_s^n|^2 ds\right)^{q/2} + E\left(\int_0^{\tau_k} dK_s^n\right)^q \\ & \leq C\left(E \sup_{0 \leq t \leq \tau_k} |Y_t^1|^q + E \sup_{0 \leq t \leq \tau_k} |\bar{Y}_t|^q + E\left(\int_0^{\tau_k} d|V|_s\right)^q + E\left(\int_0^{\tau_k} f_n^-(s, Y_s^n, 0) ds\right)^q\right) \\ & \leq C\left(E \sup_{0 \leq t \leq \tau_k} |Y_t^1|^q + E \sup_{0 \leq t \leq \tau_k} |\bar{Y}_t|^q + (2k)^q + \left(\int_0^{\tau_k} d|V|_s\right)^q\right). \end{aligned} \quad (3.12)$$

Set $g_n(s) = f_n(s, Y_s^n, 0)$, $h_n(s) = f_n(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, 0)$. From the above, the definition of $\{\tau_k\}$ and (3.11) it follows that g_n, h_n satisfy the assumptions of Lemma 2.7 (see also Remark 2.8). Hence, for $q < 2$,

$$E \int_0^{\tau_k} |Z_s^n - Z_s|^q ds \rightarrow 0,$$

and, by stationarity of $\{\tau_k\}$, $Z^n \rightarrow Z$ in measure $\lambda \otimes P$ on $[0, T] \times \Omega$. By this and by (3.11) and (3.12),

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dK_s + \int_t^T dV_s - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad (3.13)$$

where $Y_t = \lim_{n \rightarrow \infty} Y_t^n$, $K_t = \lim_{n \rightarrow \infty} K_t^n$. It is obvious that Y is regulated and $Y_t \geq L_t$, $t \in [0, T]$. We have to show the minimality condition for K and integrability of Z and K . We know that $\sum_{t < T} (Y_t^n - L_t) \Delta^+ K_t^n = 0$. Letting $n \rightarrow \infty$ we obtain

$$\sum_{t < T} (Y_t - L_t) \Delta^+ K_t = 0.$$

Therefore to prove the minimality condition for K it suffices to show that

$$\int_0^T (Y_{t-} - \underline{L}_t) dK_t^* = 0 \quad (3.14)$$

where \underline{L}_t is defined as in Proposition 3.7. Note that

$$\int_0^T (Y_{t-}^n - \underline{L}_t) dK_t^{n,*} = \int_0^T (Y_{t-}^n - \underline{L}_t) dK_t^{n,c} + \sum_{0 < t \leq T} (Y_{t-}^n - \underline{L}_t) \Delta^- K_t^n.$$

We know that $dK^n \rightarrow dK$ in the total variation norm and that $0 \leq Y_{t-}^n - \underline{L}_t \leq Y_t^1 - \underline{L}_t$. Therefore applying the Lebesgue dominated convergence theorem we get $\int_0^T (Y_{t-} - \underline{L}_t) dK_t^c = 0$. This gives (3.14) if $\Delta^- K = 0$. If $\Delta^- K_t > 0$ for some $t \in (0, T]$, then there exists $N \in \mathbb{N}$ such that $\Delta^- K_t^n > 0$ for $n \geq N$. Hence $Y_{t-}^n = \underline{L}_t$, $n \geq N$. By Proposition 3.3 and Remark 3.5, $Y_{t-} \leq Y_{t-}^n = \underline{L}_t$, and consequently, $Y_{t-} = \underline{L}_t$. Hence

$$\sum_{t \leq T} (Y_{t-} - \underline{L}_t) \Delta^- K_t = 0,$$

so (3.14) is satisfied. This proves the minimality condition for K . Note that by (H6) the process X is of the form

$$X_t = X_0 + \int_0^t dC_s + \int_0^t H_s dB_s$$

for some $C \in \mathcal{V}^p$, $H \in \mathcal{H}$. It can be rewritten in the form

$$X_t = \xi + \int_t^T f(s, X_s, H_s) ds + \int_t^T dC'_s + \int_t^T dV_s - \int_t^T H_s dB_s$$

for some $C' \in \mathcal{V}^p$. Let (\bar{X}, \bar{Z}) be a solution of BSDE $(\xi, f + dV^+ + dC', +)$. By Proposition 3.3, $\bar{X} \geq X$, so $\bar{X} \geq L$. Note that the triple $(\bar{X}, \bar{H}, C', +)$ is a solution of RBSDE $(\xi, f + dV^+, \bar{X})$. Since $\bar{X} \geq L$, then by Proposition 3.3 for $p > 1$, $\bar{X} \geq Y$. For $p = 1$ we can not for now apply Proposition 3.3 since we do not know a priori that $Z \in \mathcal{H}^q$ for some $q > \alpha$ (see Remark 3.5). Let (\bar{X}^n, \bar{H}^n) be a solution of BSDE $(\xi, f_n + dV^+ + dC', +)$. By Proposition 3.3, $\bar{X}^n \geq \bar{X} \geq L$. Hence, by Proposition 3.3 again,

$$\bar{X}^n \geq Y^n, \quad n \geq 1. \quad (3.15)$$

In the same manner as in the proof of (3.13) we show that $\bar{X}_t^n \searrow \tilde{X}_t$, $t \in [0, T]$, $\bar{H}^n \rightarrow \tilde{H}$ in measure $\lambda \otimes P$ on $[0, T] \times \Omega$, and

$$\tilde{X}_t = \xi + \int_t^T f(s, \tilde{X}_s, \tilde{H}_s) ds + \int_t^T dC'_s + \int_t^T dV_s^+ - \int_t^T \tilde{H}_s dB_s.$$

Since $\bar{Y} \leq \tilde{X} \leq \bar{X}^1$, it follows that $\tilde{X} \in \mathcal{S}^q$, $q \in (0, 1)$. By [1, Lemma 3.1], $\tilde{Z} \in \mathcal{H}^q$, $q \in (0, 1)$. Therefore by Proposition 3.3 and Remark 3.5, $\tilde{X} = \bar{X}$. By this and (3.15), $\bar{X} \geq Y$ for $p = 1$. By [12, Lemma 4.2, Proposition 4.3] we have integrability of Y , Z and K for $p \geq 1$. \square

4 Penalization method for reflected BSDEs

We assume that the barrier L has regulated trajectories. We consider approximation of the solution of RBSDE $(\xi, f + dV, L)$ by a modified penalization method of the form

$$\begin{aligned} Y_t^n = & \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T dV_s - \int_t^T Z_s^n dB_s \\ & + n \int_t^T (Y_s^n - L_s)^- ds + \sum_{t \leq \sigma_{n,i} < T} (Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-, \quad t \in [0, T] \end{aligned} \quad (4.1)$$

with specially defined arrays of stopping times $\{\{\sigma_{n,i}\}\}$ exhausting right-side jumps of L and V . We define $\{\{\sigma_{n,i}\}\}$ inductively. We first set $\sigma_{1,0} = 0$ and

$$\sigma_{1,i} = \inf\{t > \sigma_{1,i-1} : \Delta^+ L_s < -1 \text{ or } \Delta^+ V_s < -1\} \wedge T, \quad i = 1, \dots, k_1$$

for some $k_1 \in \mathbb{N}$. Next, for $n \in \mathbb{N}$ and given array $\{\{\sigma_{n,i}\}\}$ we set $\sigma_{n+1,0} = 0$ and

$$\sigma_{n+1,i} = \inf\{t > \sigma_{n+1,i-1} : \Delta^+ L_s < -1/(n+1) \text{ or } \Delta^+ V_s < -1/(n+1)\} \wedge T$$

for $i = 1, \dots, j_{n+1}$, where j_{n+1} is chosen so that $P(\sigma_{n+1, j_{n+1}} < T) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\sigma_{n+1, i} = \sigma_{n+1, j_{n+1}} \vee \sigma_{n, i-j_{n+1}}, \quad i = j_{n+1} + 1, \dots, k_{n+1}, \quad k_{n+1} = j_{n+1} + k_n.$$

Since $\Delta^+ L_s < -1/n$ or $\Delta^+ V_s < -1/n$ implies that $\Delta^+ L_s < -1/(n+1)$ or $\Delta^+ V_s < -1/(n+1)$, it follows from our construction that

$$\bigcup_i [[\sigma_{n, i}]] \subset \bigcup_i [[\sigma_{n+1, i}]] \quad n \in \mathbb{N}. \quad (4.2)$$

Moreover, on each interval $(\sigma_{n, i-1}, \sigma_{n, i}]$, $i = 1, \dots, k_n + 1$, where $\sigma_{n, k_n+1} = T$, the pair (Y^n, Z^n) is a solution of the classical generalized BSDEs of the form

$$\begin{aligned} Y_t^n &= L_{\sigma_{n, i}} \vee (Y_{\sigma_{n, i}+}^n + \Delta^+ V_{\sigma_{n, i}}) + \int_t^{\sigma_{n, i}} f(s, Y_s^n, Z_s^n) ds + \int_t^{\sigma_{n, i}} dV_s \\ &\quad + n \int_t^{\sigma_{n, i}} (Y_s^n - L_s)^- ds - \int_t^{\sigma_{n, i}} Z_s^n dB_s, \quad t \in (\sigma_{n, i-1}, \sigma_{n, i}] \end{aligned} \quad (4.3)$$

and $Y_0^n = L_0 \vee (Y_{0+}^n + \Delta^+ V_0)$, $n \in \mathbb{N}$. Observe that (4.1) can be written in the following shorter form

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T dV_s + \int_t^T dK_s^n - \int_t^T Z_s^n dB_s, \quad (4.4)$$

where

$$K_t^n = n \int_0^t (Y_s^n - L_s)^- ds + \sum_{t \leq \sigma_{n, i} < T} (Y_{\sigma_{n, i}+}^n + \Delta^+ V_{\sigma_{n, i}} - L_{\sigma_{n, i}})^- =: K_t^{n,*} + K_t^{n,d}.$$

For similar approximation scheme see [15]. As compared with the usual penalization method, the term K^n includes the purely jumping part $K^{n,d}$ consisting of right jumps. If the processes L, V are right-continuous then $K^n = K^{n,*}$, so (4.1) (or, equivalently, (4.4)) reduces to the usual penalization scheme. Note that if Y is a limit of increasing sequence $\{Y^n\}$ of càdlàg solutions of BSDEs, then by the monotone convergence theorem for BSDEs (see, e.g., [21]), Y is also càdlàg. On the other hand, if L is a regulated process, then in general the solution Y need not be càdlàg. Therefore the usual penalization equations have to be modified by adding right jumps corrections.

Theorem 4.1. *Let (Y^n, Z^n) , $n \in \mathbb{N}$, be a solution of (4.1).*

- (i) *Assume that $p > 1$ and (H1)–(H6) are satisfied. Then $Y_t^n \nearrow Y_t$, $t \in [0, T]$, and for any $\gamma \in [1, 2)$,*

$$E \left(\int_0^T |Z_s^n - Z_s|^\gamma ds \right)^{p/2} \rightarrow 0, \quad (4.5)$$

where (Y, Z, K) is unique solution of RBSDE($\xi, f + dV, L$). Moreover, if $\Delta^- K_t = 0$ for $t \in (0, T]$, then (4.5) hold true with $\gamma = 2$.

- (ii) *Assume that $p = 1$ and (H1)–(H5), (H6*), (Z) are satisfied. Then $Y_t^n \nearrow Y_t$, $t \in [0, T]$, and for any $\gamma \in [1, 2)$ and $r \in (0, 1)$,*

$$E \left(\int_0^T |Z_s^n - Z_s|^\gamma ds \right)^{r/2} \rightarrow 0, \quad (4.6)$$

where (Y, Z, K) is a unique solution of RBSDE($\xi, f + dV, L$). If $\Delta^- K_t = 0$ for $t \in (0, T]$, then (4.6) hold true with $\gamma = 2$.

Proof. Let $p \geq 1$. Without loss of generality we may assume that $\mu = 0$. Let (Y^n, Z^n) , $n \in \mathbb{N}$ be a solution of (4.1). By Proposition 3.3, $Y_t^n \leq Y_t^{n+1}$, $t \in [0, T]$, $n \in \mathbb{N}$. The rest of the proof we divide into 3 steps.

Step 1. We first show that for $n \in \mathbb{N}$ the triple (Y^n, Z^n, K^n) is a solution of RBSDE $(\xi, f + dV, L^n)$ with $L^n = L - (Y^n - L)^-$. Note that $Y_t^n \geq L_t^n$, $t \in [0, T]$. Indeed, if $Y_t^n \geq L_t$ then $Y_t^n \geq L_t^n$, and if $Y_t^n < L_t$ then $Y_t^n \geq Y_t^n = L_t^n$. Moreover,

$$\begin{aligned} \int_0^T (Y_{s-}^n - L_{s-}^n) dK_s^{n,*} &= n \int_0^T (Y_s^n - L_s^n)(Y_s^n - L_s)^- ds \\ &= n \int_0^T (Y_s^n - L_s)^+(Y_s^n - L_s)^- ds = 0 \end{aligned}$$

and

$$\begin{aligned} \sum_{s < T} (Y_s^n - L_s^n) \Delta^+ K_s^n &= \sum_{\sigma_{n,i} < T} (Y_{\sigma_{n,i}}^n - L_{\sigma_{n,i}}^n) (Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- \\ &= \sum_{\sigma_{n,i} < T} (Y_{\sigma_{n,i}}^n - L_{\sigma_{n,i}})^+ (Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- = 0. \end{aligned}$$

Indeed, suppose that

$$\sum_{\sigma_{n,i} < T} (Y_{\sigma_{n,i}}^n - L_{\sigma_{n,i}})^+ (Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- \neq 0. \quad (4.7)$$

Then there is $1 \leq i \leq k_n$ such that $Y_{\sigma_{n,i}}^n - L_{\sigma_{n,i}} > 0$ and $Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}} < 0$. By the last inequality and (4.4), $\Delta^+ Y_{\sigma_{n,i}}^n = \Delta^+ K_{\sigma_{n,i}}^n - \Delta^+ V_{\sigma_{n,i}} = -(Y_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- - \Delta^+ V_{\sigma_{n,i}}$. Hence $Y_{\sigma_{n,i}}^n = L_{\sigma_{n,i}}$, which contradicts (4.7).

Step 2. We now show that $Y_t := \sup_{n \geq 1} Y_t^n$, $t \in [0, T]$, is a regulated process satisfying condition (d) of Definition 3.1 and that (Y, Z, K) has the desired integrability properties. To this end, we first prove that if $p > 1$ then (4.5) holds true, and if $p = 1$, then there exists a stationary sequence of stopping times $\{\tau_k\}$ such that for any $\gamma \in [1, 2)$ and $r \in (0, 1)$,

$$E \left(\int_0^{\tau_k} |Z_s^n - Z_s|^\gamma ds \right)^{r/2} \rightarrow 0.$$

To show this we will use [12, Lemma 4.2]. Let $p > 1$. Then by (H6) there exists a process $X \in (\mathcal{M}_{loc} + \mathcal{V}^p) \cap \mathcal{S}^p$ such that $X \geq L$ and $\int_0^T f^-(s, X_s, 0) ds \in L^p$. If $p = 1$ then by (H6*) there exists X of class (D) such that $X \in \mathcal{M}_{loc} + \mathcal{V}^1$, $X \geq L$ and $\int_0^T f^-(s, X_s, 0) ds \in L^1$. Since the Brownian filtration has the representation property, there exist processes $H \in \mathcal{M}_{loc}$ and $C \in \mathcal{V}^p$ such that

$$X_t = X_T - \int_t^T dC_s - \int_t^T H_s dB_s, \quad t \in [0, T],$$

which can be rewritten in form

$$X_t = \xi + \int_t^T f(s, X_s, H_s) ds + \int_t^T dV_s + \int_t^T dK'_s - \int_t^T dA'_s - \int_t^T H_s dB_s$$

for some $A', K' \in \mathcal{V}^{+,p}$ with $p \geq 1$. Let (\bar{X}^n, \bar{H}^n) be a solution of the BSDE

$$\begin{aligned}\bar{X}_t^n &= \xi + \int_t^T f(s, \bar{X}_s^n, \bar{H}_s^n) ds + \int_t^T dV_s + \int_t^T dK'_s - \int_t^T \bar{H}_s^n dB_s \\ &\quad + \sum_{t \leq \sigma_{n,i} < T} (\bar{X}_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-, \quad t \in [0, T].\end{aligned}$$

By Proposition 3.3 and Remark 3.5, $\bar{X}^n \geq X \geq L$, so we may rewrite the above equation in the form

$$\begin{aligned}\bar{X}_t^n &= \xi + \int_t^T f(s, \bar{X}_s^n, \bar{H}_s^n) ds + \int_t^T dV_s + \int_t^T dK'_s + n \int_t^T (\bar{X}_s^n - L_s)^- ds \\ &\quad + \sum_{t \leq \sigma_{n,i} < T} (\bar{X}_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- - \int_t^T \bar{H}_s^n dB_s, \quad t \in [0, T].\end{aligned}$$

By Proposition 3.3 and Remark 3.5, $\bar{X}^n \geq Y^n$. Also note that

$$\begin{aligned}(\bar{X}_{\sigma_{n,i}+}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- &\leq (X_{\sigma_{n,i}+} + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- \\ &= (\Delta^+ X_{\sigma_{n,i}} + \Delta^+ V_{\sigma_{n,i}} + X_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- \leq (\Delta^+ X_{\sigma_{n,i}} + \Delta^+ V_{\sigma_{n,i}}) \\ &\leq \Delta^+ |C|_{\sigma_{n,i}} + \Delta^+ |V|_{\sigma_{n,i}}.\end{aligned}$$

Let (\tilde{X}, \tilde{H}) be a solution of the BSDE

$$\begin{aligned}\tilde{X}_t &= \xi + \int_t^T f(s, \tilde{X}_s, \tilde{H}_s) ds + \int_t^T dV_s + \int_t^T dK'_s + n \int_t^T (\tilde{X}_s - L_s)^- ds \\ &\quad + \int_t^T d|C|_s + \int_t^T d|V|_s - \int_t^T \tilde{H}_s dB_s, \quad t \in [0, T].\end{aligned}$$

The pair (\tilde{X}, \tilde{H}) does not depend on n , because by Proposition 3.3 and Remark 3.5, $\tilde{X} \geq \bar{X}^n$, so the term involving n on the right-hand side of the above equation equals zero. By the last inequality we also have $\tilde{X} \geq Y^n$. Thus all the assumptions of [12, Lemma 4.2] are satisfied. Applying [12, Lemma 4.2] we get

$$\begin{aligned}E(K_T^n)^p + E\left(\int_0^T |Z_s^n|^2 ds\right)^{p/2} &\leq CE\left(\sup_{t \leq T} (|Y_t^1|^p + |\tilde{X}_t|^p) + \left(\int_0^T d|V|_s\right)^p\right. \\ &\quad \left.+ \left(\int_0^T |f^-(s, \tilde{X}_s, 0)| ds\right)^p + \left(\int_0^T \tilde{X}_s^+ ds\right)^p + \left(\int_0^T |f(s, 0, 0)| ds\right)^p\right) \quad (4.8)\end{aligned}$$

if $p > 1$, which means that $\{Z^n\}$ is bounded in \mathcal{H}^p . If $p = 1$ then by [12, Lemma 4.2], for any $q \in (0, 1)$ we have

$$\begin{aligned}E\left(\int_0^T |Z_s^n|^2 ds\right)^{q/2} &\leq CE\left(\sup_{t \leq T} (|Y_t^1|^q + |\tilde{X}_t|^q) + \left(\int_0^T |f(s, 0, 0)| ds\right)^q\right. \\ &\quad \left.+ \left(\int_0^T |f^-(s, \tilde{X}_s, 0)| ds\right)^q + \left(\int_0^T \tilde{X}_s^+ ds\right)^q + \left(\int_0^T d|V|_s\right)^q\right). \quad (4.9)\end{aligned}$$

We next check that the assumption of Theorem 2.9 are satisfied. We know that Y^n is of class (D), $Z^n \in \mathcal{H}$, $K^n \in \mathcal{V}^+$ and $t \mapsto f(t, Y_t^n, Z_t^n) \in L^1(0, T)$. Since V is a

finite variation process and $A^n = -V$, we have $A^n \leq A^{n+1}$ and $E|A^n|_T < \infty$ for $n \in \mathbb{N}$, i.e. assumption (a) is satisfied. Let $\tau, \sigma \in \mathcal{T}$ be stopping times such that $\sigma \leq \tau$. By the Lebesgue dominated convergence theorem, $\lim_{n \rightarrow \infty} \int_{\sigma}^{\tau} (Y_s - Y_s^n) dV_s^* = 0$ and $\lim_{n \rightarrow \infty} \sum_{\sigma \leq s < \tau} (Y_s - Y_s^n) \Delta^+ V_s = 0$. Since we know that $\int_{\sigma}^{\tau} (Y_s - Y_s^n) dK_s^{n,*} + \sum_{\sigma \leq s < \tau} (Y_s - Y_s^n) \Delta^+ K_s^n \geq 0$, it follows that

$$\liminf_{n \rightarrow \infty} \left(\int_{\sigma}^{\tau} (Y_s - Y_s^n) d(K_s^n - A_s^n)^* + \sum_{\sigma \leq s < \tau} (Y_s - Y_s^n) \Delta^+ (K_s^n - A_s^n) \right) \geq 0,$$

i.e. (b) is satisfied. It is easy to see that $\Delta^- K_t^n = 0$ for $n \in \mathbb{N}$ and $t \in [0, T]$, so (c) is satisfied. Let $\bar{y} = Y^1$ and $\underline{y} = \tilde{X}$. Then $\bar{y}, \underline{y} \in \mathcal{V}^1 + \mathcal{M}_{loc}$, \bar{y}, \underline{y} are of class (D) and

$$E \int_0^T f^+(s, \bar{y}_s, 0) ds + E \int_0^T f^-(s, \underline{y}_s, 0) ds < \infty.$$

Since we already have shown that $\bar{y}_t \leq Y_t^n \leq \underline{y}_t$, $t \in [0, T]$, condition (d) is satisfied. Condition (e) follows from (H3), whereas (f) is satisfied by the very definition of Y . Thus all the assumptions of Theorem 2.9 are satisfied. Therefore Y is regulated and there exist $K \in \mathcal{V}^+$, $Z \in \mathcal{H}$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

Furthermore, $Z^n \rightarrow Z$ in measure $\lambda \otimes P$, which when combined with (4.8) and (4.9) implies that if $p > 1$ then $Z \in \mathcal{H}^p$ and (4.5) is satisfied, whereas if $p = 1$, then $Z \in \mathcal{H}^q$ for $q \in (0, 1)$ and there exists a stationary sequence $\{\tau_k\}$ such that

$$E \int_0^{\tau_k} |Z_s^n - Z_s|^\gamma ds \rightarrow 0, \quad \gamma \in [1, 2). \quad (4.10)$$

We will show that

$$\sup_{n \geq 1} E \left(\int_0^T |f(s, Y_s^n, Z_s^n)| ds \right)^p + E \left(\int_0^T |f(s, Y_s, Z_s)| ds \right)^p < \infty. \quad (4.11)$$

If $p > 1$ then by (H1),

$$\begin{aligned} & E \left(\int_0^T |f(s, Y_s^n, Z_s^n)| ds \right)^p \\ & \leq C_p \left(\left(\int_0^T |f(s, \tilde{X}_s, 0)| ds \right)^p + \left(\int_0^T |f(s, Y_s^1, 0)| ds \right)^p + E \left(\int_0^T |Z_s^n|^2 ds \right)^{p/2} \right). \end{aligned}$$

If $p = 1$ then by (Z),

$$E \int_0^T |f(s, Y_s^n, Z_s^n)| ds \leq \gamma E \int_0^T (g_s + |Y_s^n| + |Z_s^n|)^\alpha ds + E \int_0^T |f(s, Y_s^n, 0)| ds.$$

By Hölder's inequality and (H2),

$$\begin{aligned} & \gamma E \int_0^T (g_s + |Y_s^n| + |Z_s^n|)^\alpha ds + E \int_0^T |f(s, Y_s^n, 0)| ds \\ & \leq E \left(\int_0^T |Z_s^n|^2 ds \right)^{\alpha/2} + \gamma E \int_0^T (g_s + |\tilde{X}_s| + |Y_s^1|)^\alpha ds \\ & \quad + E \int_0^T |f(s, Y_s^1, 0)| + |f(s, \tilde{X}_s, 0)| ds. \end{aligned}$$

By Fatou's lemma, (4.8), (4.9) we have (4.11), which when combined with integrability of Y, K implies that $K \in \mathcal{V}^{p,+}$.

Step 3. We show that the triple (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f + dV, L)$. By (4.11), $\sup_{n \geq 1} EK_T^n < \infty$, so $\{n \int_0^T (Y_s^n - L_s)^- ds\}$ is bounded in L^1 . Therefore, up to a subsequence, $(Y_t^n - L_t)^- \rightarrow 0$ P -a.s. for a dense subset of t . Hence $Y_t \geq L_t$ for a dense subset of t . Consequently, $Y_{t+} \geq L_{t+}$ for every $t \in [0, T)$. In fact, $Y_t \geq L_t$ for every $t \in [0, T)$. Indeed, if $\Delta^+(L_t + V_t) \geq 0$ for some $t \in [0, T)$ then

$$Y_t + V_t = -\Delta^+(Y_t + V_t) + Y_{t+} + V_{t+} \geq Y_{t+} + V_{t+} \geq L_{t+} + V_{t+} \geq L_t + V_t$$

whereas if $\Delta^+(L_t + V_t) < 0$ for some $t \in [0, T)$ then $t \in \bigcup_i [\sigma_{n,i}]$ for sufficiently large n , which implies that $\Delta^+K_t^n = (Y_{t+}^n - L_t + \Delta^+V_t)^-$. Suppose that $Y_t^n < L_t$ for some t . Since $\Delta^+(Y_t + V_t) = -\Delta^+K_t^n$, we then have

$$Y_{t+}^n - L_t + \Delta^+V_t < Y_{t+}^n - Y_t^n + \Delta^+V_t = -(Y_{t+}^n - L_t + \Delta^+V_t)^-,$$

which ... contradiction. Thus $Y_t^n \geq L_t$ for every $t \in [0, T)$, and hence $Y_t \geq L_t$ for $t \in [0, T)$. Consequently,

$$Y_t \geq L_t \mathbf{1}_{\{t < T\}} + \xi \mathbf{1}_{\{t = T\}}, \quad t \in [0, T].$$

Now we are going to show the minimality condition for K . Since $Y_t + \int_0^t f(s, Y_s, Z_s) ds - V_t$, $t \in [0, T]$, is a supermartingale, it follows from the properties of the Snell envelope that

$$Y_t \geq \text{ess sup}_{\tau \in \Gamma_t} E \left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t \right). \quad (4.12)$$

If $p > 1$ then by Proposition 3.9 and the definition of L^n , for $t \in [0, T]$ we have

$$\begin{aligned} Y_t^n &= \text{ess sup}_{\tau \in \Gamma_t} E \left(\int_t^\tau f(s, Y_s^n, Z_s^n) ds + \int_t^\tau dV_s + L_\tau^n \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t \right) \\ &\leq \text{ess sup}_{\tau \in \Gamma_t} E \left(\int_t^\tau f(s, Y_s^n, Z_s^n) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t \right). \end{aligned}$$

Observe that by (4.5), (4.11) and the assumptions on f ,

$$E \int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds \rightarrow 0.$$

By Lemma 3.15,

$$Y_t \leq \text{ess sup}_{\tau \in \Gamma_t} E \left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\tau < T} + \xi \mathbf{1}_{\tau = T} | \mathcal{F}_t \right).$$

By the above inequality and (4.12),

$$Y_t = \text{ess sup}_{\tau \in \Gamma_t} E \left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\tau < T} + \xi \mathbf{1}_{\tau = T} | \mathcal{F}_t \right).$$

By Corollary 3.8 we have the minimality condition for K . Hence the triple (Y, Z, K) is a solution of $\text{RBSDE}(\xi, f + dV, L)$ on $[0, T]$.

Consider now the case $p = 1$. Since $Y^1 \leq Y^n \leq Y$, $n \geq 1$, by (H2) we have

$$f(t, Y_t, 0) \leq f(t, Y_t^n, 0) \leq f(t, Y_t^1, 0), \quad t \in [0, T].$$

Set

$$\sigma_k = \inf\{t \geq 0; \int_0^t |f(s, Y_s^1, 0)| ds + \int_0^t |f(s, Y_s, 0)| ds \geq k\} \wedge T.$$

It is clear that $\{\sigma_k\}$ is stationary. We may assume that $\sigma_k = \tau_k$. By Proposition 3.9 and the definition of L^n ,

$$\begin{aligned} Y_t^n &= \operatorname{ess\,sup}_{\tau_k \geq \tau, \tau \in \Gamma_t} E\left(\int_t^\tau f(s, Y_s^n, Z_s^n) ds + \int_t^\tau dV_s + L_\tau^n \mathbf{1}_{\{\tau < \tau_k\}} + Y_{\tau_k}^n \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t\right) \\ &\leq \operatorname{ess\,sup}_{\tau_k \geq \tau, \tau \in \Gamma_t} E\left(\int_t^\tau f(s, Y_s^n, Z_s^n) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < \tau_k\}} + Y_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t\right). \end{aligned}$$

Observe that by (4.10), the definition of σ_k and the assumptions on f ,

$$E \int_0^{\tau_k} |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds \rightarrow 0.$$

By Lemma 3.15,

$$Y_t \leq \operatorname{ess\,sup}_{\tau_k \geq \tau, \tau \in \Gamma_t} E\left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < \tau_k\}} + Y_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t\right).$$

By the above inequality and (4.12),

$$Y_t = \operatorname{ess\,sup}_{\tau_k \geq \tau, \tau \in \Gamma_t} E\left(\int_t^\tau f(s, Y_s, Z_s) ds + \int_t^\tau dV_s + L_\tau \mathbf{1}_{\{\tau < \tau_k\}} + Y_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t\right).$$

By Corollary 3.8 we have the minimality condition for K on $[0, \tau_k]$, and by stationarity of $\{\tau_k\}$ also on $[0, T]$. Therefore (Y, Z, K) is the solution of RBSDE $(\xi, f + dV, L)$ on $[0, T]$. \square

5 Appendix. Itô's formula for processes with regulated trajectories

We consider an \mathbb{F} -adapted process X with regulated trajectories of the form

$$X_t = X_t^* + \sum_{s < t} \Delta^+ X_s, \quad t \in [0, T], \quad (5.1)$$

where X^* is an \mathbb{F} -adapted semimartingale with càdlàg trajectories and

$$\sum_{s < T} |\Delta^+ X_s| < \infty, \quad P\text{-a.s.}$$

(note that $\Delta^- X_s = \Delta X_s^*$).

Theorem 5.1 ([7, 18]). *Let $(X_t)_{t \leq T}$ be an adapted process with regulated trajectories of the form (5.1), and let f be a real function of class C^2 . Then the process $(f(X_t))_{t \leq T}$ also has the form (5.1). More precisely, for every $t \in [0, T]$,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-}) dX_s^* + \frac{1}{2} \int_0^t f''(X_{s-}) d[X^*]_s^c + J_t^- + J_t^+,$$

where $J_t^- = \sum_{s \leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta^- X_s\}$, $J_t^+ = \sum_{s < t} \{f(X_{s+}) - f(X_s)\}$.

Note that the two sums defining J^- and J^+ are absolutely convergent, and that J^- is a càdlàg adapted process, whereas J^+ is càglàd adapted. Indeed,

$$|J^-|_t \leq C_1 \sum_{s \leq t} |\Delta^- X_s|^2 = C_1 \sum_{s \leq t} |\Delta X_s^*|^2, \quad P\text{-a.s.}$$

and

$$|J^+|_t \leq C_2 \sum_{s < t} |\Delta^+ X_s|, \quad P\text{-a.s.},$$

where C_1, C_2 are random variables defined by $C_1 = (1/2) \sup_{x \in [-M, M]} |f''(x)|$ and $C_2 = \sup_{x \in [-M, M]} |f'(x)|$, where $M = \sup_{s \leq T} |X_s|$ (note that $M < \infty$ P -a.s.) We include the proof of Theorem 5.1 for completeness of our presentation.

Proof. Set $X_t^+ = X_{t+}$, $t \leq T$. Clearly

$$X_t^+ = \Delta^+ X_t + X_t = X_t^* + \sum_{s \leq t} \Delta^+ X_s, \quad t \leq T.$$

Hence X^+ is a semimartingale. By Itô's formula for semimartingales,

$$\begin{aligned} f(X_t^+) &= f(X_0) + \int_0^t f'(X_{s-}^+) dX_s^+ + \frac{1}{2} \int_0^t f''(X_{s-}^+) d[X^*]_s^c \\ &\quad + \sum_{s \leq t} \{f(X_s^+) - f(X_{s-}^+) - f'(X_{s-}^+) \Delta X_s^+\}. \end{aligned}$$

Observe that $X_{s-}^+ = X_{s-}$, $f(X_s^+) = f(X_s) + f(X_{s+}) - f(X_s)$ and $\Delta X_s^+ = \Delta^+ X_s + \Delta^- X_s$. Hence

$$\begin{aligned} f(X_t^+) &= f(X_0) + \int_0^t f'(X_{s-}) dX_s^* + \sum_{s \leq t} f'(X_{s-}) \Delta^+ X_s + \frac{1}{2} \int_0^t f''(X_{s-}^+) d[X^*]_s^c \\ &\quad + \sum_{s \leq t} \{f(X_{s+}) - f(X_{s-}) - f'(X_{s-}) (\Delta^+ X_s + \Delta^- X_s)\} \\ &= f(X_0) + \int_0^t f'(X_{s-}) dX_s^* + \frac{1}{2} \int_0^t f''(X_{s-}^+) d[X^*]_s^c \\ &\quad + \sum_{s \leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta^- X_s\} + \sum_{s \leq t} \{f(X_{s+}) - f(X_s)\}. \quad (5.2) \end{aligned}$$

Subtracting $f(X_{t+}) - f(X_t)$ from both sides of (5.2) we obtain the desired formula. \square

Corollary 5.2. *Let $X = (X^1, \dots, X^d)$ be an adapted d -dimensional process with regulated trajectories of the form (5.1) and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a function of class C^2 . Then the process $(f(X_t))_{t \leq T}$ also has the form (5.1). Moreover, for every $t \in [0, T]$,*

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_{s-}) dX_s^{i,*} \\ &\quad + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) d[X^{i,*}, X^{j,*}]_s^c + J_t^- + J_t^+, \end{aligned}$$

where $J_t^- = \sum_{s \leq t} \{f(X_s) - f(X_{s-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{s-}) \Delta^- X_s^i\}$, $J_t^+ = \sum_{s < t} \{f(X_{s+}) - f(X_s)\}$.

Corollary 5.3. *Let X^1, X^2 be two adapted processes with regulated trajectories of the form (5.1). Then*

$$\begin{aligned} X_t^1 X_t^2 &= X_0^1 X_0^2 + \int_0^t X_{s-}^1 dX_s^{2,*} + \int_0^t X_{s-}^2 dX_s^{1,*} + [X^{1,*}, X^{2,*}]_t \\ &\quad + \sum_{s < t} (X_{s+}^1 X_{s+}^2 - X_s^1 X_s^2), \quad t \in [0, T]. \end{aligned}$$

Corollary 5.4. *Let $X = (X^1, \dots, X^d)$ be an adapted d -dimensional process with regulated trajectories of the form (5.1). Then for all $p \geq 1$ and $t \in [0, T]$,*

$$\begin{aligned} |X_t|^p &= |X_0|^p + p \int_0^t |X_{s-}|^{p-1} \langle \text{s\hat{g}n}(X_{s-}), dX_s^* \rangle + p \sum_{s < t} |X_s|^{p-1} \langle \text{s\hat{g}n}(X_s), \Delta^+ X_s \rangle \\ &\quad + \frac{p}{2} \int_0^t |X_s|^{p-2} \mathbf{1}_{\{X_s \neq 0\}} \{ (2-p) (1 - \langle \text{s\hat{g}n}(X_s), Q_s^{X^*} \text{s\hat{g}n}(X_s) \rangle) + (p-1) \} d[X^*]_s^c \\ &\quad + L_t \mathbf{1}_{\{p=1\}} + J_t^-(p) + J_t^+(p), \end{aligned}$$

where Q^{X^*} denotes the Radon-Nikodym derivative $d[[X^*]]^c/d[X^*]^c$, $(L_t)_{t \leq T}$ is an adapted increasing continuous process such that $L_0 = 0$, and

$$J_t^-(p) = \sum_{s \leq t} \{ |X_s|^p - |X_{s-}|^p - p |X_{s-}|^{p-1} \langle \text{s\hat{g}n}(X_{s-}), \Delta^- X_s \rangle \}, \quad t \in [0, T]$$

and

$$J_t^+(p) = \sum_{s < t} \{ |X_{s+}|^p - |X_s|^p - p |X_s|^{p-1} \langle \text{s\hat{g}n}(X_s), \Delta^+ X_s \rangle \}, \quad t \in [0, T]$$

are adapted increasing processes with càdlàg and càglàd trajectories, respectively.

Proof. We follow the proof of [1, Lemma 2.2] (see also the proof of [12, Proposition 2.1]). The formula is an easy consequence of Corollary 5.2 in the case where $p \geq 2$. Assume that $p \in [1, 2)$ and for $\epsilon > 0$ set $u_\epsilon(x) = (|x|^2 + \epsilon^2)^{1/2}$, $x \in \mathbb{R}^d$. Clearly, u_ϵ^p is a smooth approximation of $|\cdot|^p$. It is easy to check that $\frac{\partial u_\epsilon^p}{\partial x_i}(x) = p u_\epsilon^{p-2}(x) x_i$ for $i = 1, \dots, d$, $x \in \mathbb{R}^d$, and

$$\frac{\partial^2 u_\epsilon^p}{\partial x_i \partial x_j}(x) = p(p-2) u_\epsilon^{p-4}(x) x_i x_j + p u_\epsilon^{p-2}(x) \mathbf{1}_{\{i=j\}}, \quad i, j = 1, \dots, d, x \in \mathbb{R}^d.$$

By Corollary 5.2,

$$\begin{aligned} u_\epsilon^p(X_t) &= u_\epsilon^p(X_0) + p \int_0^t u_\epsilon^{p-2}(X_{s-}) \langle X_{s-}, dX_s^* \rangle + p \sum_{s < t} u_\epsilon^{p-2}(X_s) \langle X_s, \Delta^+ X_s \rangle \\ &\quad + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \{ p(p-2) u_\epsilon^{p-4}(X_s) X_s^i X_s^j + p u_\epsilon^{p-2}(X_s) \mathbf{1}_{\{i=j\}} \} d[X^{i,*}, X^{j,*}]_s^c \\ &\quad + \sum_{s \leq t} \{ u_\epsilon^p(X_s) - u_\epsilon^p(X_{s-}) - p u_\epsilon^{p-2}(X_{s-}) \langle X_{s-}, \Delta^- X_s \rangle \} \\ &\quad + \sum_{s < t} \{ u_\epsilon^p(X_{s+}) - u_\epsilon^p(X_s) - p u_\epsilon^{p-2}(X_s) \langle X_s, \Delta^+ X_s \rangle \} \\ &=: u_\epsilon^p(X_0) + I_t^{1,\epsilon} + I_t^{2,\epsilon} + I_t^{3,\epsilon} + I_t^{4,\epsilon} + I_t^{5,\epsilon}, \end{aligned}$$

where using (5.1) we separated $I^{2,\epsilon}$ from right side jumps J^+ . Since $u_\epsilon^p(x) \rightarrow |x|^p$ $x \in \mathbb{R}^d$, it is clear that

$$u_\epsilon^p(X_t) \rightarrow |X_t|^p, \quad t \in [0, T], \quad P\text{-a.s.} \quad (5.3)$$

Moreover, the convergence $u_\epsilon^{p-2}(x)x \rightarrow |x|^{p-1}\text{s\hat{g}n}(x)$, $x \in \mathbb{R}^d$ implies that

$$I_t^{1,\epsilon} \xrightarrow{P} p \int_0^t |X_{s-}|^{p-1} \langle \text{s\hat{g}n}(X_{s-}), dX_s^* \rangle, \quad t \in [0, T] \quad (5.4)$$

and, by (5.1), that

$$I_t^{2,\epsilon} \rightarrow p \sum_{s < t} |X_{s-}|^{p-1} \langle \text{s\hat{g}n}(X_s), \Delta^+ X_s \rangle, \quad t \in [0, T], \quad P\text{-a.s.} \quad (5.5)$$

Similarly,

$$I_t^{5\epsilon} \rightarrow J_t^+, \quad t \in [0, T], \quad P\text{-a.s.} \quad (5.6)$$

On the other hand, using the identity $u_\epsilon^{p-2}(x) = u_\epsilon^{p-4}(x)|x|^2 + \epsilon^2 u_\epsilon^{p-4}(x)$ we get

$$\begin{aligned} I_t^{3,\epsilon} &= \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \{p(p-2)u_\epsilon^{p-4}(X_s)X_s^i X_s^j + pu_\epsilon^{p-4}(X_s)|X_s|^2 \mathbf{1}_{\{i=j\}}\} d[X^{i,*}, X^{j,*}]_s^c \\ &\quad + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t p\epsilon^2 u_\epsilon^{p-4}(X_s) \mathbf{1}_{\{i=j\}} d[X^{i,*}, X^{j,*}]_s^c \\ &= \frac{1}{2} p \sum_{i=1}^d \sum_{j=1}^d \int_0^t (2-p)u_\epsilon^{p-4}(X_s)|X_s|^2 (\mathbf{1}_{\{i=j\}} - \frac{X_s^i}{|X_s|} \frac{X_s^j}{|X_s|}) \mathbf{1}_{\{X_s \neq 0\}} d[X^{i,*}, X^{j,*}]_s^c \\ &\quad + \frac{1}{2} p \sum_{i=1}^d \int_0^t (p-1)u_\epsilon^{p-4}(X_s)|X_s|^2 d[X^{i,*}]_s^c + \frac{p}{2} \sum_{i=1}^d \int_0^t \epsilon^2 u_\epsilon^{p-4}(X_s) d[X^{i,*}]_s^c \\ &= \frac{p}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t (2-p)u_\epsilon^{p-4}(X_s)|X_s|^2 (\mathbf{1}_{\{i=j\}} - \frac{X_s^i}{|X_s|} \frac{X_s^j}{|X_s|}) Q_s^{X^*}(i, j) \mathbf{1}_{\{X_s \neq 0\}} d[X^*]_s^c \\ &\quad + \frac{p}{2} \int_0^t (p-1)u_\epsilon^{p-4}(X_s)|X_s|^2 d[X^*]_s^c + \frac{p}{2} \int_0^t \epsilon^2 u_\epsilon^{p-4}(X_s) d[X^*]_s^c \\ &=: I_t^{6,\epsilon} + I_t^{7,\epsilon} + I_t^{8,\epsilon}. \end{aligned}$$

Since $Q_s^{X^*}$ is a symmetric non-negative matrix with a trace equal to 1,

$$\begin{aligned} &\sum_{i=1}^d \sum_{j=1}^d (\mathbf{1}_{\{i=j\}} - \frac{X_s^i}{|X_s|} \frac{X_s^j}{|X_s|}) Q_s^{X^*}(i, j) \mathbf{1}_{\{X_s \neq 0\}} \\ &= (1 - \langle \text{s\hat{g}n}(X_s), Q_s^{X^*} \text{s\hat{g}n}(X_s) \rangle) \mathbf{1}_{\{X_s \neq 0\}} \geq 0, \quad s \in [0, T]. \end{aligned} \quad (5.7)$$

By this and the fact that $|x|/u_\epsilon(x) \nearrow \mathbf{1}_{\{x \neq 0\}}$, $x \in \mathbb{R}^d$, it follows that for $t \in [0, T]$,

$$I_t^{6,\epsilon} \nearrow \frac{1}{2} p \int_0^t (2-p)|X_s|^{p-2} (1 - \langle \text{s\hat{g}n}(X_s), Q_s^{X^*} \text{s\hat{g}n}(X_s) \rangle) \mathbf{1}_{\{X_s \neq 0\}} d[X^*]_s^c \quad (5.8)$$

P -a.s. Similarly,

$$I_t^{7,\epsilon} \nearrow \frac{1}{2}p \int_0^t (p-1)|X_s|^{p-2} \mathbf{1}_{\{X_s \neq 0\}} d[X^*]_s^c, \quad t \in [0, T], \quad P\text{-a.s.} \quad (5.9)$$

From (5.3)–(5.9) we deduce that there is a process B with regulated trajectories such that $I_t^{4,\epsilon} + I_t^{8,\epsilon} \rightarrow B_t$ in probability P for $t \in [0, T]$, and

$$\begin{aligned} |X_t|^p &= |X_0|^p + p \int_0^t |X_{s-}|^{p-1} \langle \text{s\grave{g}n}(X_{s-}), dX_s^* \rangle + p \sum_{s < t} |X_s|^{p-1} \langle \text{s\grave{g}n}(X_s), \Delta^+ X_s \rangle \\ &\quad + \frac{1}{2}p \int_0^t |X_s|^{p-2} \mathbf{1}_{\{X_s \neq 0\}} \{ (2-p)(1 - \langle \text{s\grave{g}n}(X_s), Q_s^{X^*} \text{s\grave{g}n}(X_s) \rangle) + (p-1) \} d[X^*]_s^c \\ &\quad + B_t + J_t^+(p). \end{aligned} \quad (5.10)$$

Since the function u_ϵ^p is convex, the processes $I^{8,\epsilon}, I^{4,\epsilon}$ are increasing. It follows that B is also increasing. Moreover, $B_0 = 0$ and $B_t = L_t + \sum_{s \leq t} \Delta^- B_s + \sum_{s < t} \Delta^+ B_s$, where L is the continuous part of B . Comparing the jumps of the left and right-hand side of (5.10) we obtain that $\sum_{s \leq t} \Delta^- B_s = J_t^-(p)$ and $\sum_{s < t} \Delta^+ B_s = 0$. Moreover, it follows from the arguments from the proof of [1, Lemma 2.2] that $L = 0$ in the case where $p > 1$. \square

Corollary 5.5. *Let $X = (X^1, \dots, X^d)$ be an adapted d -dimensional process with regulated trajectories of the form (5.1). Then for all $p \in [1, 2]$ and $t \in [0, T]$,*

$$\begin{aligned} |X_t|^p + \frac{p(p-1)}{2} \int_0^t |X_s|^{p-2} \mathbf{1}_{\{X_s \neq 0\}} d[X^*]_s^c + J_T^-(p) - J_t^-(p) + J_T^+(p) - J_t^+(p) \\ \leq |X_T|^p + p \int_t^T |X_{s-}|^{p-1} \langle \text{s\grave{g}n}(X_{s-}), dX_s^* \rangle + p \sum_{t \leq s < T} |X_s|^{p-1} \langle \text{s\grave{g}n}(X_s), \Delta^+ X_s \rangle. \end{aligned}$$

Proof. Follows from Corollary 5.4 and (5.7). \square

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